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# INEQUALITIES ON TIME SCALES

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**Ph.D. Thesis**  
- Abstract -

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**-2008-**

## INTRODUCTION

This thesis is mainly concerned with the study of time scales. The theory on time scales was introduced by Stefan Hilger in his Ph.D. thesis [24], supervised by Bernd Aulbach, in order to unify discrete analysis and continuous analysis.

The present paper's goal is to bring some generalizations of some classical inequalities from Mathematical Analysis, especially from Convex Analysis. In the same time, there are presented some generalizations of the results and methods used to obtain some inequalities and it is highlighted the antagonistic problem of the continuous case and the discrete case. We also study some aspects attached to convexity on time scales and we study the properties of some functions defined on such type of sets.

An essential philosophical element of this paper is represented by the special link between the continuous and the discrete, and also, by the fundamental difference between those two notions.

From the structural point of view, the paper is organized in 6 chapters, and we shall present them subsequently, pointing out the important results.

## 1. THE CALCULUS ON TIME SCALES

In Chapter 1, we present the main notions connected to the time scales calculus. For some functions defined on time scales, we define the standard derivatives and integrals (delta and nabla), and also the combined diamond- $\alpha$  derivative and integral. The main results of this mathematical branch are presented here, together with some new results, very useful among the other chapters, such as the calculus of delta and nabla integrals on time scales of the following type  $\{a_0, a_1, \dots, a_n, \dots, b\}$  with  $\lim_{n \rightarrow \infty} a_n = b$ , (see [10]).

**Proposition 1.** (Proposition 1.3). *Let  $a, b \in \mathbb{T}$  and  $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ .*

*(iv) If  $(a_n)_{n \in \mathbb{N}}$  is a nondecreasing convergent sequence such that  $\lim_{n \rightarrow \infty} a_n = b$  and  $[a, b]_{\mathbb{T}} = \{a_0, a_1, \dots, a_n, \dots, b\}$ , then*

$$\int_a^b f(t) \Delta t = \lim_{n \rightarrow \infty} \sum_{i=0}^n f(a_i)(a_{i+1} - a_i).$$

*If  $(a_n)_{n \in \mathbb{N}}$  is a nonincreasing convergent sequence with  $\lim_{n \rightarrow \infty} a_n = b$  and  $[a, b]_{\mathbb{T}} = \{b, \dots, a_n, \dots, a_1, a_0\}$ , then*

$$\int_a^b f(t) \Delta t = \lim_{n \rightarrow \infty} \sum_{i=0}^n f(a_{i+1})(a_i - a_{i+1}).$$

Important examples of time scales are presented here and they will received a lot of consideration among the paper. Obviously, this examples contain the real numbers set  $\mathbb{R}$  and the integer numbers set  $\mathbb{Z}$ ,

but also the set of the multiples of a real number  $h > 0$  and the set of integer powers of a real number  $q > 1$ , including 0 (this time scales leads to what is called  $q$ -calculus). Other interesting examples are the reunions of closed disjoint intervals (those are used often in population dynamics) or even more “exotic” time scales, such as the Cantor set. On each of those time scales, we illustrate with examples the main operators used in time scale calculus (the forward jump operator  $\sigma$  and the backward jump operator  $\rho$ ), but also the calculus techniques used in the other chapters. The theorems that give the main properties of the operators used on time scales are also presented here. Thus, we outline the properties that are common to the usual derivatives and integrals, but also the ones that are peculiar to the new defined notions, that will guide us to a close understanding of this domain.

A special case is the diamond- $\alpha$  derivative and integral, since the combined  $\diamond_\alpha$  derivative is not a dynamic derivative (see Example 1.8 and the relation (1.4)).

## 2. TIME SCALE BETWEEN THE DISCRETE AND THE CONTINUOUS CASE

In Chapter 2, we establish some results starting with the idea that a time scale is a set obtained from the set of real numbers, by eliminating some open intervals, or it is obtained from other time scales, by adding or extracting intervals or isolated points. Thus, in the case of monotone functions, we can determine relations between the delta, nabla and diamond- $\alpha$  integrals on time scales and the real numbers set. The affine functions play a very important role in real analysis and the same is expected to occur on time scales. That is the reason why the study of the integrals of an affine function on any time scale can offer very powerful tools for obtaining new inequalities.

In Section 2, we introduce a new, original notion, **the measure of graininess** between two points of a time scale.

**Definition 1.** (Definition 2.1). *Let  $\mathbb{T}$  a time scale and  $a, b \in \mathbb{T}$ , finite points. One defines the measure of graininess between  $a$  and  $b$  to be the function  $G : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}_+$  given by*

$$G(a, b) = \sum_{a \leq t < b} \frac{\mu(t)^2}{2} = \sum_{a < t \leq b} \frac{\nu(t)^2}{2}.$$

Further, we present some examples where this function interferes, and also some properties of this function.

**Proposition 2.** (Proposition 2.1). *Let  $\mathbb{T}$  a time scale,  $a, b \in \mathbb{T}$  and the measure of graininess  $G : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}_+$ . Then*

$$(1) \quad G(a, b) = G(a, x) + G(x, b), \text{ for all } x \in [a, b]_{\mathbb{T}},$$

$$(2) \quad \int_a^b t \diamond_{\alpha} t = \frac{b^2 - a^2}{2} + (1 - 2\alpha)G(a, b), \text{ for all } \alpha \in [0, 1],$$

$$(3) \quad G(a, b) = 0 \text{ if and only if } [a, b] = [a, b]_{\mathbb{T}}.$$

Also, in this Section we compute the integrals  $\int_a^b t \Delta t$ ,  $\int_a^b t \nabla t$  and  $\int_a^b t \diamond_{\alpha} t$  with respect to  $\int_a^b t dt$  and the measure of graininess  $G(a, b)$ , (see [10]).

Another element of novelty appears in Section 3, where we define the convex functions on time scales. This notion generalizes and includes the notion of convex function defined on real numbers set and also the notion of convex sequence.

**Definition 2.** (Definition 2.2). *A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called convex on  $I_{\mathbb{T}}$ , if*

$$(4) \quad f(\lambda t + (1 - \lambda)s) \leq \lambda f(t) + (1 - \lambda)f(s),$$

for all  $t, s \in I_{\mathbb{T}}$  and all  $\lambda \in [0, 1]$  such that  $\lambda t + (1 - \lambda)s \in I_{\mathbb{T}}$ .

The function  $f$  is strictly convex on  $I_{\mathbb{T}}$  if the inequality (4) is strict for every distinct points  $t, s \in I_{\mathbb{T}}$  and  $\lambda \in (0, 1)$ .

The function  $f$  is concave (respectively strictly concave) on  $I_{\mathbb{T}}$ , if  $-f$  is convex (respectively strictly convex).

A function that is both convex and concave on  $I_{\mathbb{T}}$  is called affine on  $I_{\mathbb{T}}$ .

The connection between the convexity of a function and the monotony of the delta and nabla derivatives is presented in Theorem 2.1 and its nabla analogue, and also, the connection with secondary delta and nabla derivatives (Corollary 2.3).

**Theorem 1.** (Theorem 2.1). *Let  $f : I_{\mathbb{T}} \rightarrow \mathbb{R}$  a delta diferentiabile function on  $I_{\mathbb{T}}^{\kappa}$ . If  $f^{\Delta}$  is nondecreasing (nonincreasing) on  $I_{\mathbb{T}}^{\kappa}$ , then  $f$  is convex (concave) on  $I_{\mathbb{T}}$ .*

A result that establish the strong connection between the convexity on real numbers set and on a time scale is Theorem 2.4.

**Theorem 2.** (Theorem 2.4). *A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is convex on  $I_{\mathbb{T}} = I \cap \mathbb{T}$  if and only if it exists a convex function  $\tilde{f} : I \rightarrow \mathbb{R}$  such that  $\tilde{f}(t) = f(t)$ , for all  $t \in I_{\mathbb{T}}$ .*

Also, the continuity of a convex function on an open interval of a time scale is also proved here. A notion specific to continuous functions, the subdifferential, is extended to convex functions defined on time scales and there are presented some of its properties.

The new results from this Chapter are published in the papers [10] and [13].

### 3. THE HERMITE–HADAMARD INEQUALITY

The third Chapter is dedicated to the Hermite–Hadamard Inequality. The first Section deals with a positive-weights version.

**Theorem 3.** (Theorem 3.1). *Let  $\mathbb{T}$  be a time scale,  $a, b \in \mathbb{T}$  and  $m, M \in \mathbb{R}$ . Let  $f : [m, M] \rightarrow \mathbb{R}$  a continuous and convex function,  $g \in C([a, b]_{\mathbb{T}}, [m, M])$  and  $w \in C([a, b]_{\mathbb{T}}, \mathbb{R})$  such that  $w(t) \geq 0$  for all  $t \in [a, b]_{\mathbb{T}}$  and  $\int_a^b w(t) \diamond_{\alpha} t > 0$ . Then*

$$(5) \quad \begin{aligned} f(x_{g,w,\alpha}) &\leq \frac{1}{\int_a^b w(t) \diamond_{\alpha} t} \int_a^b f(g(t))w(t) \diamond_{\alpha} t \\ &\leq \frac{M - x_{g,w,\alpha}}{M - m} f(m) + \frac{x_{g,w,\alpha} - m}{M - m} f(M), \end{aligned}$$

where  $x_{g,w,\alpha} = \int_a^b g(t)w(t) \diamond_{\alpha} t / \int_a^b w(t) \diamond_{\alpha} t$ .

If the weights are equal, a generalization for monotone functions emerges (Theorem 3.3).

The second Section handles the complete variant with weights of the Hermite–Hadamard Inequality. More precisely, the  $\alpha$ -Steffensen–Popoviciu and  $\alpha$ -Hermite–Hadamard weights are defined, and it is further proved that the first verify the left-hand side of the inequality (5), while the latter verify the right-hand side of the inequality (5).

**Definition 3.** (Definition 3.1). *Let  $\mathbb{T}$  be a time scale and  $g : \mathbb{T} \rightarrow \mathbb{R}$  a continuous function. The continuous function  $w : \mathbb{T} \rightarrow \mathbb{R}$  is a  $\alpha$ -Steffensen–Popoviciu weight for  $g$  on  $[a, b]_{\mathbb{T}}$  (briefly,  $\alpha$ -SP weight) if*

$$(6) \quad \int_a^b w(t) \diamond_{\alpha} t > 0 \quad \text{and} \quad \int_a^b f(g(t))^+ w(t) \diamond_{\alpha} t \geq 0$$

for all  $f : [m, M] \rightarrow \mathbb{R}$  continuous and convex functions, where  $m = \inf_{t \in [a, b]_{\mathbb{T}}} g(t)$  and  $M = \sup_{t \in [a, b]_{\mathbb{T}}} g(t)$ .

Some results belonging to *T. Popoviciu* are generalized to exemplify this category of weights. It is of interest that also negative weights are admissible on the definition interval, if some final positivity conditions are satisfied (The complete weighted Jensen’s inequality). Theorem 3.5 states that the  $\alpha$ -Steffensen–Popoviciu weights are also  $\alpha$ -Hermite–Hadamard weights. Thus, the right-hand side of the inequality is verified by all the weights satisfying the left-hand side of the inequality.

An extension of the Hermite–Hadamard inequality for a certain type of functions that have a weak symmetry property, the  $(g, w, \alpha)$ -symmetric functions, appears in Section 3.

**Definition 4.** (Definition 3.3). *Let  $\mathbb{T}$  be a time scale,  $a, b \in \mathbb{T}$ ,  $m, M \in \mathbb{R}$ ,  $\alpha \in [0, 1]$ ,  $g \in C([a, b]_{\mathbb{T}}, [m, M])$  and  $w \in C(\mathbb{T}, \mathbb{R})$  a  $\alpha$ -Steffensen–Popoviciu weight for  $g$  over  $[a, b]_{\mathbb{T}}$ . The function  $f : [m, M] \rightarrow \mathbb{R}$  is  $(g, w, \alpha)$ -symmetric over  $[a, b]_{\mathbb{T}}$  if*

- (i)  $\frac{M-x_{g,w,\alpha}}{M-m} f(m) + \frac{x_{g,w,\alpha}-M}{M-m} f(M) = f(x_{g,w,\alpha});$
- (ii)  $\int_a^b f(g(t))w(t)\diamond_{\alpha}t = f(x_{g,w,\alpha}) \int_a^b w(t)\diamond_{\alpha}t,$

where  $x_{g,w,\alpha} = \left( \int_a^b g(t)w(t)\diamond_{\alpha}t \right) / \left( \int_a^b w(t)\diamond_{\alpha}t \right).$

If  $\alpha = 1/2$ , some of the above-mentioned results give similar results to the classical ones known from the real numbers set.

The new results of this Chapter are published in: [9], [10], [14] and [17].

#### 4. APPLICATIONS OF THE HERMITE–HADAMARD INEQUALITY

Chapter 4 overhears some of many applications of the Hermite–Hadamard inequality for time scales, such as Hölder’s inequality and Ky Fan’s inequality, and it presents some generalizations of this inequalities, in the time scale context.

Using the complete weighted Jensen’s Theorem, we can get an improved version of Hölder’s inequality.

**Theorem 4.** (Hölder’s inequality). *Let  $\mathbb{T}$  a time scale,  $a < b \in \mathbb{T}$ ,  $f, g \in C([a, b]_{\mathbb{T}}, [0, +\infty))$  and  $w \in C([a, b]_{\mathbb{T}}, \mathbb{R})$  such that  $wg^q$  is an  $\alpha$ -SP weight for  $fg^{-q/p}$  on  $[a, b]_{\mathbb{T}}$ , where  $p$  and  $q$  are Hölder conjugates (that is,  $1/p + 1/q = 1$  and  $p > 1$ ). Then we have*

$$(7) \quad \int_a^b w(t)f(t)g(t)\diamond_{\alpha}t \leq \left( \int_a^b w(t)f^p(t)\diamond_{\alpha}t \right)^{1/p} \left( \int_a^b w(t)g^q(t)\diamond_{\alpha}t \right)^{1/q}.$$

In the first Section, we consider a function that will guide us to the discovery of an improvement of the Hölder’s inequality, using its monotonicity properties. More exactly, with the above notations, we define the function  $h : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ , given by

$$(8) \quad h(x, y) = \left( \int_x^y w(t)f^p(t)\diamond_{\alpha}t \right)^{1/p} \left( \int_x^y w(t)g^q(t)\diamond_{\alpha}t \right)^{1/q} - \int_x^y w(t)f(t)g(t)\diamond_{\alpha}t.$$

Some consequences of Hölder’s inequality, otherwise hard to obtain, are also included.

In Section 2 we define the arithmetic, geometric and harmonic generalized weighted means of a function defined on a time scale.

**Definition 5.** (Definition 4.1). *Let  $x : \mathbb{T} \rightarrow \mathbb{R}_+$  a continuous and positive function and  $w$  a weight. We define:*

(i) the generalized arithmetic weighted mean of the function  $x$  on the interval  $[a, b]_{\mathbb{T}}$  of  $w$  weight:

$$(9) \quad A_{[a,b]}(x, w, \alpha) = \frac{\int_a^b w(t)x(t)\diamond_{\alpha}t}{\int_a^b w(t)\diamond_{\alpha}t};$$

(ii) the generalized geometric weighted mean of the function  $x$  on the interval  $[a, b]_{\mathbb{T}}$  of  $w$  weight:

$$(10) \quad G_{[a,b]}(x, w, \alpha) = \exp\left(\frac{\int_a^b w(t)\log(x(t))\diamond_{\alpha}t}{\int_a^b w(t)\diamond_{\alpha}t}\right);$$

(iii) the generalized harmonic weighted mean of the function  $x$  on the interval  $[a, b]_{\mathbb{T}}$  of  $w$  weight:

$$(11) \quad H_{[a,b]}(x, w, \alpha) = \frac{\int_a^b w(t)\diamond_{\alpha}t}{\int_a^b w(t)/x(t)\diamond_{\alpha}t}.$$

Using these definitions, we prove two versions of Ky Fan's inequality, which uses both sides of the Hermite–Hadamard inequality.

**Theorem 5.** (Theorem 4.4). *Let  $x : \mathbb{T} \rightarrow [m, M]$  a continuous and positive function such that  $0 < m \leq x(t) \leq M \leq \gamma/2$ ,  $\gamma > 0$  and  $w : \mathbb{T} \rightarrow \mathbb{R}$  an  $\alpha$ -Steffensen–Popoviciu weight for  $x$  on  $[a, b]_{\mathbb{T}}$ . Then*

$$(12) \quad \begin{aligned} \frac{A_{[a,b]}(x, w, \alpha)}{G_{[a,b]}(x, w, \alpha)} &\geq \left(\frac{A_{[a,b]}(x, w, \alpha)}{G_{[a,b]}(x, w, \alpha)}\right)^{M^2/(\gamma-M)^2} \geq \frac{A_{[a,b]}(\gamma - x, w, \alpha)}{G_{[a,b]}(\gamma - x, w, \alpha)} \\ &\geq \left(\frac{A_{[a,b]}(x, w, \alpha)}{G_{[a,b]}(x, w, \alpha)}\right)^{m^2/(\gamma-m)^2} \geq 1. \end{aligned}$$

Moreover,

$$\frac{A_{[a,b]}(x, w, \alpha)}{A_{[a,b]}(\gamma - x, w, \alpha)} \geq \frac{G_{[a,b]}(x, w, \alpha)}{G_{[a,b]}(\gamma - x, w, \alpha)}.$$

The next theorem uses the right side of the Hermite–Hadamard inequality.

**Theorem 6.** *Let  $x : [a, b]_{\mathbb{T}} \rightarrow [m, M]$  a continuous and positive function such that  $0 < m \leq x(t) \leq M \leq \frac{\gamma}{2}$ ,  $\gamma > 0$  and  $w : \mathbb{T} \rightarrow \mathbb{R}$  an  $\alpha$ -Hermite–Hadamard weight for  $x$  on  $[a, b]_{\mathbb{T}}$ . Then*

$$\begin{aligned}
(13) \quad & \left( \frac{\gamma - m}{m \frac{m^2}{(\gamma - m)^2}} \right)^{(M - A_{[a,b]}(x, w, \alpha)) / (M - m)} \\
& \cdot \left( \frac{\gamma - M}{M \frac{m^2}{(\gamma - m)^2}} \right)^{(A_{[a,b]}(x, w, \alpha) - m) / (M - m)} \cdot G_{[a,b]}(x, w, \alpha)^{m^2 / (\gamma - m)^2} \\
& \leq G_{[a,b]}(\gamma - x, w, \alpha) \\
& \leq \left( \frac{\gamma - m}{M \frac{M^2}{(\gamma - M)^2}} \right)^{(M - A_{[a,b]}(x, w, \alpha)) / (M - m)} \\
& \cdot \left( \frac{\gamma - M}{M \frac{M^2}{(\gamma - M)^2}} \right)^{(A_{[a,b]}(x, w, \alpha) - m) / (M - m)} \cdot G_{[a,b]}(x, w, \alpha)^{M^2 / (\gamma - M)^2}.
\end{aligned}$$

The new results of this Chapter are published in [11] and [19].

## 5. OSTROWSKI AND IYENGAR TYPE INEQUALITIES

It is well-known that the deviance from the integral mean in the Hermite–Hadamard inequality is estimated by the the Ostrowski’s inequality for the left-hand side and by the Iyengar’s inequality for the right-hand side. In Chapter 5, some generalizations of Ostrowski type inequalities are proved, Section 1 containing a weighted version, while Section 2 develops two continuous functions variants.

**Theorem 7.** (Weighted Ostrowski’s Inequality). *Let  $a, b, s, t \in \mathbb{T}$  and  $a < b$ , while  $p, q \in \mathbb{R}$ ,  $1/p + 1/q = 1$  and  $p > 1$ .*

- (i) *If  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  is a delta derivable function and  $w \in C_{rd}([a, b]_{\mathbb{T}})$  a positive weight, then*

$$\begin{aligned}
(14) \quad & \left| f(t) - \frac{1}{\int_a^b w(s) \Delta s} \int_a^b w(s) f(\sigma(s)) \Delta s \right| \\
& \leq \frac{\|f^\Delta\|_\infty}{\int_a^b w(s) \Delta s} \int_a^b w(s) |\sigma(s) - t| \Delta s \\
& \leq \frac{\|f^\Delta\|_\infty}{\int_a^b w(s) \Delta s} \left\{ \begin{aligned} & \left( \int_a^b |\sigma(s) - t|^p \Delta s \right)^{\frac{1}{p}} \left( \int_a^b (w(s))^q \Delta s \right)^{\frac{1}{q}}; \\ & \|w\|_\infty \left[ \left( t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} - G(a, t) + G(t, b) \right]; \\ & \|w\|_\infty \left( \frac{b - \sigma(a)}{2} + \left| t - \frac{b + \sigma(a)}{2} \right| \right). \end{aligned} \right.
\end{aligned}$$



(ii) If  $f : [a, b] \rightarrow \mathbb{R}$  is a nabla derivable function and  $w \in C_{ld}([a, b]_{\mathbb{T}})$  is a positive weight then

$$\begin{aligned}
 (15) \quad & \left| f(t) - \frac{1}{\int_a^b w(s) \nabla s} \int_a^b w(s) f(\rho(s)) \nabla s \right| \\
 & \leq \frac{\|f^\nabla\|_\infty}{\int_a^b w(s) \nabla s} \int_a^b w(s) |\rho(s) - t| \nabla s \\
 & \leq \frac{\|f^\nabla\|_\infty}{\int_a^b w(s) \nabla s} \begin{cases} \left( \int_a^b |\rho(s) - t|^p \nabla s \right)^{\frac{1}{p}} \left( \int_a^b (w(s))^q \nabla s \right)^{\frac{1}{q}}; \\ \|w\|_\infty \left[ \left( t - \frac{a+b}{2} \right)^2 + \frac{(b-a)^2}{4} + G(a, t) - G(t, b) \right]; \\ \|w\|_\infty \left( \frac{b-\rho(a)}{2} + \left| t - \frac{b+\rho(a)}{2} \right| \right). \end{cases}
 \end{aligned}$$

The Ostrowski type theorems for two continuous functions are the following.

**Theorem 8.** (Theorem 5.6). *Let  $f, g : \mathbb{T} \rightarrow \mathbb{R}$  be two continuous functions over  $\mathbb{T}$ , with bounded delta and nabla derivatives. (i.e.  $\|f^\Delta\|_\infty, \|g^\Delta\|_\infty, \|f^\nabla\|_\infty, \|g^\nabla\|_\infty < \infty$ ). Hence*

$$\begin{aligned}
 & \left| f(t)g(t) - \frac{1}{2(b-a)} \left[ g(t) \int_a^b f(s) \diamond_{\alpha} s + f(t) \int_a^b g(s) \diamond_{\alpha} s \right] \right| \\
 & \leq \frac{1}{2} \{ \alpha [ \|g(t)\| \|f^\Delta\|_\infty + \|f(t)\| \|g^\Delta\|_\infty ] \\
 & \quad + (1-\alpha) [ \|g(t)\| \|f^\nabla\|_\infty + \|f(t)\| \|g^\nabla\|_\infty ] \} \\
 & \quad \cdot \left[ \frac{1}{4} + \left( \frac{t - \frac{a+b}{2}}{b-a} \right)^2 + (1-2\alpha) \frac{G(t, b) - G(a, t)}{(b-a)^2} \right] (b-a),
 \end{aligned}$$

for all  $t \in \mathbb{T}$ , where  $G$  is the measure of graininess.

**Theorem 9.** (Theorem 5.7). *Let  $f, g : \mathbb{T} \rightarrow \mathbb{R}$  be continuous functions over  $\mathbb{T}$ , with bounded delta and nabla derivatives. Then*

$$\begin{aligned}
 & \left| f(t)g(t) - \frac{1}{b-a} \left[ g(t) \int_a^b f(s) \diamond_{\alpha} s \right. \right. \\
 & \quad \left. \left. + f(t) \int_a^b g(s) \diamond_{\alpha} s \right] + \frac{1}{b-a} \int_a^b f(s)g(s) \diamond_{\alpha} s \right| \\
 & \leq \frac{1}{b-a} [ \alpha \|f^\Delta\|_\infty + (1-\alpha) \|f^\nabla\|_\infty ] \\
 & \quad \cdot [ \alpha \|g^\Delta\|_\infty + (1-\alpha) \|g^\nabla\|_\infty ] \int_a^b |t-s|^2 \diamond_{\alpha} s,
 \end{aligned}$$

for all  $t \in \mathbb{T}$ ,  $a \leq t \leq b$ .

Section 3 focuses on the proof of a diamond- $\alpha$  variant of the Iyengar's inequality, for diamond- $\alpha$  integrable functions, without any further delta, nabla or diamond- $\alpha$  differentiability constraints as in most variants of the Iyengar's inequality, including the original one.

**Theorem 10.** (Iyengar's Inequality for time scales). *Let  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  be a function from  $C_{ld}([a, b]_{\mathbb{T}}) \cap C_{rd}([a, b]_{\mathbb{T}})$ . Suppose that*

$$\frac{a+b}{2} - \frac{f(b)-f(a)}{2M}, \frac{a+b}{2} + \frac{f(b)-f(a)}{2M} \in \mathbb{T}$$

and for all  $t \in [a, b]_{\mathbb{T}}$ , exists  $M > 0$  such that

$$(16) \quad |f(t) - f(a)| \leq M(t-a) \quad \text{and} \quad |f(t) - f(b)| \leq M(b-t).$$

Hence,

$$(17) \quad \left| \int_a^b f(t) \diamond_{\alpha} t - \frac{f(a)+f(b)}{2}(b-a) \right| \leq \frac{M(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M} + (1-2\alpha)MG(a,b).$$

This Section also includes the generalized version of this inequality.

**Theorem 11.** (Generalized Iyengar's inequality for time scales). *Let  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  be a function from  $C_{ld} \cap C_{rd}$  over  $[a, b]_{\mathbb{T}}$ . Suppose that for all  $t \in [a, b]_{\mathbb{T}}$  exist  $M > m$  such that*

$$(18) \quad m \leq \frac{f(t)-f(a)}{t-a} \leq M \quad \text{and} \quad m \leq \frac{f(t)-f(b)}{b-t} \leq M$$

and

$$\frac{a+b}{2} - \frac{f(b)-f(a) - \frac{M+m}{2}(b-a)}{M-m},$$

$$\frac{a+b}{2} + \frac{f(b)-f(a) - \frac{M+m}{2}(b-a)}{M-m} \in \mathbb{T}.$$

Then

$$(19) \quad \left| \int_a^b f(t) \diamond_{\alpha} t - \frac{f(a)+f(b)}{2}(b-a) - (1-2\alpha)G(a,b) \right|$$

$$\leq \frac{[f(b)-f(a) - m(b-a)][M(b-a) - f(b) + f(a)]}{2(M-m)} + (1-2\alpha)G(a,b).$$

Section 4 is concerned about delta and nabla variants of Steffensen's inequalities, useful in obtaining other Iyengar-type inequalities.

**Theorem 12.** (Steffensen's inequalities for delta integrals). *Let  $g : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  a delta integrable function such that  $\lambda = \int_a^b g(t) \Delta t \in (0, b-a)$ . The following conditions are equivalent:*

- (i)  $0 \leq \int_a^x g(t) \Delta t \leq x-a$  and  $0 \leq \int_x^b g(t) \Delta t \leq b-x$ , for all  $x \in [a, b]_{\mathbb{T}}$ ;

(ii)  $\int_a^{a+\lambda} f(t)\Delta t \leq \int_a^b f(t)g(t)\Delta t \leq \int_{b-\lambda}^b f(t)\Delta t$ , for all nondecreasing functions  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$ .

The thesis also presents the nabla variant of this Theorem. As a consequence of these theorems, we get the following result.

**Theorem 13.** (Theorem 5.14). *Let  $\alpha \in [0, 1]$  and  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  a diamond- $\alpha$  derivable and diamond- $\alpha$  integrable function such that*

$$\frac{a+b}{2} - \frac{f(b) - f(a)}{2M}, \frac{a+b}{2} + \frac{f(b) - f(a)}{2M} \in \mathbb{T}$$

and

$$\frac{f(t) - f(a)}{t - a}, \frac{f(b) - f(t)}{b - t} \in [-M, M],$$

for all  $t \in [a, b]_{\mathbb{T}}$ . Then

$$\begin{aligned} (20) \quad & \frac{(f(b) - f(a))^2}{4M} - \frac{M(b-a)^2}{4} - 2(1-2\alpha)MG \left( \frac{a+b}{2} + \frac{f(b) - f(a)}{2M}, b \right) \\ & \leq \int_a^b f(t)\diamond_{\alpha} t - \frac{f(a) + f(b)}{2}(b-a) \\ & \leq \frac{M(b-a)^2}{4} - \frac{(f(b) - f(a))^2}{4M} \\ & \quad - 2(1-2\alpha)MG \left( a, \frac{a+b}{2} - \frac{f(b) - f(a)}{2M} \right). \end{aligned}$$

During this Chapter the  $G$  function (graininess measure), defined in Chapter 1, has been shown very useful.

The new results of this Chapter are published in [8] and [18].

## 6. MULTIDIMENSIONAL TIME SCALES

Chapter 6 follows the generalization of the derivatives and the integrals on  $n$ -dimensional time scales. In Section 1, we define the partial dynamic derivatives delta, nabla and diamond- $\alpha$ .

Then, in the second Section, we define the notion of generalized ‘‘homogenous segment’’, that will allow the introduction of the notion of the multiple diamond- $\alpha$  Darboux integral and then the multiple diamond- $\alpha$  Riemann integral and we prove that the two notions are similar.

**Definition 6.** (Definition 6.3). *The function  $f$  is called Darboux  $\diamond_{\alpha}$ -integrable on  $[a, b]_{\mathbb{T}}$  if  $L(f) = U(f)$ . We call it the diamond- $\alpha$  Darboux integral and denote this value by  $\int_{[a, b]_{\mathbb{T}}} f(t)\diamond_{\alpha} t$ .*

**Definition 7.** (Definition 6.4). *The function  $f$  is called Riemann  $\diamond_{\alpha}$ -integrable on  $[a, b]_{\mathbb{T}}$  if it exists a real number  $I$  with the following property: for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $P \in \mathcal{P}(a, b)$  implies*

$|S(f, P, \xi, \alpha) - I| < \varepsilon$  for any possible choice of the vectors system  $\xi$ . The number  $I$  with this property is called the Riemann diamond- $\alpha$  integral.

Some theorems that develop the calculus on  $n$ -dimensional time scales are presented, as well, in this Chapter. In Section 3, we define the notion of generalized “heterogeneous segment” and we introduce the generalized Darboux and Riemann integral. In this case, for any vector  $\alpha = (\alpha_1, \dots, \alpha_n) \in [0, 1]^n$ , the generalized diamond- $\alpha$  multiple integral is introduced iterative as a convex combination.

In Section 4 we prove a Leibniz-Newton type formula for both types of the above mentioned integrals, in the unidimensional case.

**Theorem 14.** (Theorem 6.11). *Let  $\mathbb{T}$  an unidimensional time scale and  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  a delta integrable function, in the sense of Definitions 6.4 and 6.6, that has delta antiderivatives. Then*

$$(21) \quad \int_{[a, b]_{\mathbb{T}}} f(t) \Delta t = \int_a^b f(t) \Delta t = F(b) - F(a),$$

where  $F$  is a delta antiderivative of  $f$ .

Section 5 presents a generalization of some inequalities from the other chapters, from the unidimensional time scales, to multidimensional time scales. Thus, we first prove Jensen’s inequality with multiple variables and then we get two other interesting inequalities, one of them being the generalized Čebyšev’s inequality.

**Theorem 15.** (Jensen’s inequality with multiple variables). *Let  $\mathbb{T}$  a time scale,  $a, b \in \mathbb{T}$  and  $f : S \rightarrow \mathbb{R}$  a continuous convex function. Let  $h, g_1, \dots, g_n \in C(\mathbb{T}, \mathbb{R})$  such that  $\int_a^b |h(t)| \diamond_{\alpha} t > 0$  and  $g_1([a, b]) \times \dots \times g_n([a, b]) \subset S$ . Then*

$$(22) \quad f \left( \frac{\int_a^b |h(t)| g_1(t) \diamond_{\alpha} t}{\int_a^b |h(t)| \diamond_{\alpha} t}, \dots, \frac{\int_a^b |h(t)| g_n(t) \diamond_{\alpha} t}{\int_a^b |h(t)| \diamond_{\alpha} t} \right) \leq \frac{\int_a^b |h(t)| f(g_1(t), \dots, g_n(t)) \diamond_{\alpha} t}{\int_a^b |h(t)| \diamond_{\alpha} t}.$$

In the last part of the Section, we prove Fubini’s theorem for time scales and using it, we get a version of Hermite–Hadamard inequalities for multiple integrals.

**Theorem 16.** (Hermite–Hadamard inequality for multiple integrals).

With the above notations, we have

$$\begin{aligned}
 (23) \quad & f\left(\frac{x_{g_1, w_1, \alpha_1} + \dots + x_{g_n, w_n, \alpha_n}}{n}\right) \\
 & \leq \frac{\int_{a_n}^{b_n} \dots \int_{a_1}^{b_1} w_1(t_1) \cdot \dots \cdot w_n(t_n) f\left(\frac{g_1(t_1) + \dots + g_n(t_n)}{n}\right) \diamond_{\alpha_1} t_1 \dots \diamond_{\alpha_n} t_n}{\prod_{i=1}^n \int_{a_i}^{b_i} w_i \diamond_{\alpha_i} t_i} \\
 & \leq \frac{M - \frac{x_{g_1, w_1, \alpha_1} + \dots + x_{g_n, w_n, \alpha_n}}{n}}{M - m} f(m) \\
 & \quad + \frac{\frac{x_{g_1, w_1, \alpha_1} + \dots + x_{g_n, w_n, \alpha_n}}{n} - m}{M - m} f(M).
 \end{aligned}$$

Some of the new results from this Chapter are published in [12].

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