Resumé of the Ph.D Thesis

"Nonlinear elliptic equations and applications",

by Ionică Andrei

In this thesis we are concerned with variational problems with $p(x)$ - Laplace type operator and hemivariational inequalities. All the problems studied here are containing nonlinearities and we are concerned with the existence of solutions (sometimes nonexistence, sometimes uniqueness or multiplicity).

This paper is organized in nine chapter. Chapter 1, was reserved to the Introduction. In Chapter 2 we present some history notes, some basic results on the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ and some significant recent results.

Chapter 3 is based on the paper *"Existence theorems for some classes of boundary value problems involving the p(x)-Laplacian "* published in *"Nonlinear Analysis: Modelling and Control"*. In the first section we consider the following problem

$$\begin{cases} -\Delta_{p(x)}u &= \lambda f(x,u), \quad \text{in } \Omega\,, \\ \quad u &= 0, \quad \text{on } \partial\Omega\,, \\ 0<\lambda &\le a\,, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $a>0$ is a given constant and f satisfies the following conditions:

(F_1) f is a Carathéodory function, i.e., measurable in Ω and continuous in $u \in \mathbb{R}$, with $f(x,0) \neq 0$ on a subset of Ω of positive measure;

(F_2) $\mid f(x,u) \mid \le C_1 + C_2 \mid u \mid^{q(x)-1}$, for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$, with constants $C_1 \ge 0$, $C_2 \ge 0$ and $1 < p(x) \le q(x) < p^\star(x)$, where

$$p^\star(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \ge N; \end{cases}$$

(F_3) there are constants $b_1 \ge 0, b_2 \ge 0, 1 \le \gamma < p(x) < \nu$ such that, for a.e. $x \in \Omega$ and every $u \in \mathbb{R}$,

$$f(x,u)u - \nu \int_0^u f(s,\tau)d\tau \ge -b_1 - b_2 \mid u \mid^\gamma .$$

We prove that if the function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies conditions (F_1)-(F_3) and there exists a function $\beta \in C^1(\mathbb{R},\mathbb{R})$ such that, for some constants $0 < \rho < r$, $\sigma > 0$, the following properties hold:

(β_1) $\beta(0) = \beta(r) = 0$;

(β_2) $\rho^{\sigma+1} \ge q(x)a_2\frac{||u||^{q^+}}{||u||^{q(x)}}$ and $\dfrac{\sigma+1}{q(x)}\beta(\rho) = a_1$;

(β_3) $\lim\limits_{|t|\to\infty} \beta(t) = +\infty$;

(β_4) $\beta'(t) < 0$ if and only if $t < 0$ or $\rho < t < r$,
then, for each $a > 0$, the following alternative holds:
either

(i) $a > 0$ is an eigenvalue in problem (1) with a corresponding eigenfunction $u \in W_0^{1,p(x)}(\Omega)$ located by

$$\alpha \leq -\int_\Omega \int_0^{u(x)} f(x,t)dtdx + \frac{1}{ap(x)} \|u\|^{p(x)} \leq a_1 + \alpha$$

or

(ii) one can find a positive number s with

$$\rho < s < r, \tag{2}$$

which determines an eigensolution $(u, \lambda) \in W_0^{1,p(x)}(\Omega) \times (0, a]$ of the problem (1) by the relations

$$\|u\| = s^{-\sigma/q(x)}(-\beta'(s))^{1/q(x)}, \tag{3}$$

$$\lambda^{-1} = a^{-1} + s^{(q(x)+\sigma p(x))/q(x)}(-\beta'(s))^{(q(x)-p(x))/q(x)}, \tag{4}$$

$$\alpha \leq \frac{s^{q(x)+1}}{q(x)} \|u\|^{q(x)} + \frac{\sigma+1}{q(x)} \beta(s) - \int_\Omega \int_0^{u(x)} f(x,t)dtdx + \frac{1}{ap(x)} \|u\|^{p(x)} \leq a_1 + \alpha. \tag{5}$$

In the second section of this chapter we consider problem:

$$\begin{cases} -\Delta_{p(x)}u &= \lambda \mid u \mid^{p(x)-2} u + \mid u \mid^{q(x)-2} u, \quad \text{in } \Omega\,, \\ u & = 0, \quad \text{on } \partial\Omega\,, \\ u & \not\equiv 0, \quad \text{in } \Omega\,. \end{cases} \tag{6}$$

The main result of this section is if

$$\lambda < \lambda_1(-\Delta_{p(x)}) := \inf \left\{ \int_\Omega \mid \nabla u \mid^{p(x)}; u \in W_0^{1,p(x)}(\Omega)\,, u \neq 0\,, \|u\|_{L^{p(x)}} = 1 \right\}$$

and $1 < p(x) < q(x) < p^\star(x)$, then the problem (8) has a weak solution.

In Chapter 4 we study the existence and multiplicity of solutions of the boundary value problem

$$\begin{cases} -div(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)-2}\nabla u) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We consider that f and a satisfies the following conditions:

$(F1)$ $$f \in C(\mathbb{R}, \mathbb{R}).$$

Set

$$C_+(\overline{\Omega}) = \{h; h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For $h \in C_+(\overline{\Omega})$, let

$$h^- = ess\inf_{x\in\Omega} h(x), \quad h^+ = ess\sup_{x\in\Omega} h(x).$$

$(F2)$ There exist nonnegative constants a_1, a_2 such that

$$|f(t)| \leq a_1 + a_2|t|^s, t \in \mathbb{R}.$$

where $p \in C_+(\overline{\Omega})$, $s+1 < \frac{Np(x)}{N-p(x)}$ if $N > p(x)$, and $0 \leq s < \infty$ if $N \leq p(x)$.
$(F3)$ There exist $\theta \in (0, \frac{1}{p^+})$ and $t_0 \geq 0$ such that for $|t| \geq t_0$

$$\theta t f(t) \geq F(t) > 0,$$

where $F(t) = \int_0^t f(s)ds$.
$(F4)$ F is an even function.

$(A1)$
$$a \in C(\mathbb{R}_+, \mathbb{R}).$$

$(A2)$ There exist constants b_1, $b_2 > 0$ such that

$$b_1 \leq \lim_{t\to\infty}\inf a(t) \leq \lim_{t\to\infty}\sup a(t) \leq b_2.$$

$(A3)$ $a(t)t^{\frac{p(x)-1}{p(x)}}$ is strictly increasing as t increases and $\lim_{t\to 0^+} a(t)t^{\frac{p(x)-1}{p(x)}} = 0$.
The first main result of this chapter is if the function a satisfies $(A1)$, $(A2)$, $(A3)$, f satisfies $(F1)$, $(F2)$, $(F3)$, $(F4)$, $p(x) \geq 2$ and $b_2\theta < \frac{b_1}{p^+}$, then Problem (7) possesses an unbounded sequence of weak solutions.
For the second main result we consider and the following conditions:
$(A4)$ There exist constants c_1, $c_2 > 0$ and b_1, $b_2 > 0$ such that for each $t > 0$

$$c_1 + b_1 t^{p(x)-2} \leq t^{p(x)-2} a(t^{p(x)}) \leq c_2 + b_2 t^{p(x)-2}.$$

$(F6)'$

$$\lim_{|t|\to+\infty}\sup \frac{p(x)F(t)}{|t|^{p(x)}} < \lambda_1(g(p(x))c_1 + b_1),$$

where λ_1 is the first eigenvalue of the $(-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega))$ and $g = \chi_{\{2\}}$ denotes the characteristic function of the set $\{2\}$.

$(F7)'$
$$(c_2 + g(p(x))b_2)\mu_i < \lim_{t\to 0}\inf \frac{f(t)}{t} \leq \lim_{t\to 0}\sup \frac{f(t)}{t} < (c_1 + g(p(x))b_1)\mu_{i+1}.$$

If a satisfies $(A1)$, $(A3)$, $(A4)$, f satisfies $(F1)$, $(F2)$, $(F6)'$, $(F7)'$ and $p(x) \geq 2$, then Problem (7) has at least two nontrivial solutions.

Chapter 5 is based on the paper *"Multiplicity results for nonlinear eigenvalue problems on unbounded domains"* published in *"Mathematica (Cluj)"*.
In this chapter we study the elliptic problems of the following type

$$\begin{cases} -div(a(x)|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u(x)) & \text{in } \Omega, \\ a(x)|\nabla u|^{p(x)-2}\nabla u \cdot n + b(x)|u|^{p(x)-2}u = \mu g(x,u(x)) & \text{on } \Gamma, \\ u \neq 0 & \text{in } \Omega, \end{cases} \qquad (P_{\lambda,\mu})$$

where λ, $\mu > 0$, n denotes the unit outward normal on Γ.

We also consider the following assumptions:
$(F1)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(\cdot, 0) = 0$ and

$$|f(x,s)| \leq f_0(x) + f_1(x)|s|^{r-1},$$

where $p^+ < r < \frac{p^+N}{N-p^+}$, and f_0, f_1 are measurable functions which satisfy

$$0 < f_0(x) \leq C_f w_1(x), \text{ and } 0 \leq f_1(x) \leq C_f w_1(x) \quad \text{a.e. in } \Omega,$$

$$f_0 \in L^{\frac{r}{r-1}}(\Omega; w_1^{\frac{1}{1-r}});$$

$(F2)$

$$\lim_{s\to 0} \frac{f(x,s)}{f_0(x)|s|^{p^+-1}} = 0, \quad \text{uniformly for all } x \in \Omega;$$

$(F3)$ $\limsup_{s\to+\infty} \frac{1}{f_0(x)|s^{p^+}|} F(x,s) \leq 0$ uniformly for all $x \in \Omega$ and $\max_{|s|\leq M} F(\cdot, s) \in L^1(\Omega)$ for all $M > 0$, where $F(x,u) = \int_0^u f(x,s)ds$;
$(F4)$ there exists $u_0 \in E$ such that $\int_\Omega F(x, u_0(x))dx > 0$.
$(G1)$ $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $g(\cdot, 0) = 0$ and

$$|g(x,s)| \leq g_0(x) + g_1(x)|s|^{m-1},$$

where $p^+ \leq m < p^+ \cdot \frac{N-1}{N-p^+}$, and g_0, g_1 are measurable functions satisfying

$$0 < g_0(x) \leq C_g w_2(x) \quad \text{and} \quad 0 \leq g_1(x) \leq C_g w_2(x), \quad \text{a.e. in } \Gamma,$$

$$g_0 \in L^{\frac{q}{q-1}}(\Gamma; w_2^{\frac{1}{1-q}});$$

$(G2)$

$$\lim_{s\to 0} \frac{g(x,s)}{g_0(x)|s|^{p^+-1}} = 0, \quad \text{uniformly for all } x \in \Gamma;$$

$(G3)$ $\limsup_{s\to+\infty} \frac{1}{g_0(x)|s^{p^+}|} G(x,s) < +\infty$ uniformly for all $x \in \Gamma$ and $\max_{|s|\leq M} G(\cdot, s) \in L^1(\Gamma)$ for all $M > 0$, where $G(x,u) = \int_0^u g(x,s)ds$.
We prove that if $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying conditions $(F1) - (F4)$, then there exists a non-degenerate compact interval $[a, b] \subset [0, +\infty]$ with the following properties:

i) there exists a number $\sigma_0 > 0$ such that for every $\lambda \in [a, b]$ and for every function $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfying conditions $(G1)-(G2)$, there exists $\mu_0 > 0$ such that for each $\mu \in [0, \mu_0]$, problem $(P_{\lambda,\mu})$ has at least one non-trivial solution in E with norm less than σ_0;

ii) for every $\lambda \in [a, b]$ and for every function $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ satisfying conditions $(G1)-(G3)$, there exists $\mu_1 > 0$ such that for each $\mu \in [0, \mu_1]$, problem $(P_{\lambda,\mu})$ has at least two non-trivial solutions in E.

Chapter 6 is based on the paper *"Existence and non-existence results for elliptic exterior problems with nonlinear boundary conditions"* published in *"Analele Universităţii Ovidius din Constanţa"*.

In this chapter we consider that Ω is a smooth exterior domain in $\mathbb{R}^N$, that is, Ω is the complement of a bounded domain with $C^{1,\delta}$ boundary $(0 < \delta < 1)$ and assume that $a \in L^\infty(\Omega) \bigcap C^{0,\delta}(\overline{\Omega})$ is a positive function, and $b \in L^\infty(\Omega) \bigcap C(\Omega)$ is non-negative. We study problem

$$\begin{cases} -div(a(x)|\nabla u|^{p(x)-2}\nabla u) + |u|^{q-2}u = \lambda g(x)|u|^{r-2}u & \text{in } \Omega, \\ a(x)|\nabla u|^{p(x)-2}\partial_\nu u + b(x)|u|^{p(x)-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $p \in C_+(\overline{\Omega})$, λ is a real parameter and ν is the unit vector of the outward normal on $\partial\Omega$. We assume

$(H1)$ $g \in L^\infty(\Omega) \bigcap L^{p_0}(\Omega)$, with $p_0 := p^*/(p^* - r)$, $p^+ < r < q < p^*$, is a non-negative function which is positive on a non-empty open subset of Ω, where $p^* := Np^+/(N - p^+)$;

$(H2)$ b is a continuous positive function on $\Gamma = \partial\Omega$.

We prove that $(H1)$ and $(H2)$ hold, then there exists $\lambda^* > 0$ with the following properties:

(i) if $\lambda < \lambda^*$, then Problem (8) does not have any weak solution;

(ii) if $\lambda \geq \lambda^*$, then Problem (8) has at least one weak solution u, with the properties

(a) $u \in L^\infty_{loc}(\Omega)$;

(b) $u \in C^{1,\alpha}(\Omega \cap B_R)$, $\alpha = \alpha(R) \in (0, 1)$;

(c) $u > 0$ in Ω;

(d) $u \in L^m(\Omega)$ for all $p^* \leq m < \infty$ and $\lim_{|x|\to\infty} u(x) = 0$.

In the second result of this chapter we consider condition $(H1)'$, which is exactly assumption $(H1)$, with the only exception that condition $p^+ < r < q < p^*$ is replaced by

$$p^+ < q < r < p^*.$$

In this case we prove that if the assumptions $(H1)'$ and $(H2)$ hold, then

(i) Problem (8) does not have any weak solution for any $\lambda \leq 0$;

(ii) Problem (8) has at least one weak solution u, with the properties $(a) - (d)$ for all $\lambda > 0$.

Chapter 7 is based on the paper *"Blow-up boundary solutions for a class of nonhomogeneous logistic equations"* published in *"Analele Universităţii din Craiova"*.

In this chapter we study the equations of the type $\Delta_{p(x)}u = g(x)f(u)$, where Ω is a bounded domain, g is a non-negative continuous function on Ω which is allowed to be unbounded on Ω and non-linearity f is a non-negative non-decreasing functions. We show that the equation $\Delta_{p(x)}u = g(x)f(u)$ admits a non-negative local weak solution $u \in W^{1,p(x)}_{loc}(\Omega) \cap C(\Omega)$ such that $u(x) \to \infty$ as $x \to \partial\Omega$ if $\Delta_{p(x)}w = -g(x)$ in the weak sense for some $w \in W^{1,p(x)}_0(\Omega)$ and f satisfies a generalized Keller-Osserman condition.

We find the solutions $u \in W^{1,p(x)}_{loc}(\Omega) \cap C(\Omega)$ to the problem

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) = g(x)f(u) & \text{in } \Omega, \\ u(x) \to \infty & \text{as } d(x, \partial\Omega) \to 0. \end{cases} \tag{9}$$

The function g is supposed that is non-negative, which satisfies the following condition:
for any $x_0 \in \Omega$ satisfying $g(x_0) = 0$, there exists a sub-domain

$$O \text{ with } \overline{O} \subset \Omega \text{ containing } x_0 \text{ such that } g(x) > 0 \text{ for all } x \in \partial O. \tag{10}$$

Suppose that the non-linearity f satisfies
$(F1)$ $f : [0, \infty) \to [0, \infty)$ is a non-decreasing C^1 function such that $f(0) = 0$, and
$(F2)$ $f(s) > 0$ for $s > 0$.
The growth condition on f at infinity,

$$\int_1^\infty \frac{1}{(F(t))^{1/p(x)}} dt < \infty, \quad \text{where} \quad F(t) := \int_0^t f(s) ds, \tag{11}$$

is crucial in the investigation of existence of blow-up solutions (the Keller-Osserman condition).

We prove that if $D \subseteq \mathbb{R}^N$ is a bounded domain, $g \in C(\overline{D})$ satisfied (10) on D and f satisfy the Keller-Osserman condition, then Problem

$$\begin{cases} div(|\nabla u|^{p(x)-2}\nabla u) = g(x)f(u) & \text{in } D, \\ u(x) \to \infty & \text{as } d(x, \partial D) \to 0, \end{cases} \tag{12}$$

admits a non-negative solution $u \in W^{1,p(x)}_{loc}(D) \cap C^{1,\alpha}(D)$, $0 < \alpha < 1$.

For the next result of this chapter we consider that $g \in C(\Omega)$ satisfies the condition:
There exist a sequence $\{D_k\}$ of domains such that

(1) $\overline{D}_k \subseteq D_{k+1}$; $k = 1, 2, ...$

(2) $\Omega = \bigcup_{k=1}^\infty D_k$.

(3) g satisfied condition (10) on each D_k.

If f is a function satisfying the Keller-Osserman condition and $g \in C(\Omega)$ satisfy the G-condition, then Problem (9) admits a non-negative blow-up solution, if the Dirichlet problem

$$\begin{cases} div(|\nabla w|^{p(x)-2}\nabla u) = -g(x), & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{13}$$

has a weak solution.

Chapter 8 is based on the paper *"Nonlinear hemivariational inequalities and applications to nonsmooth mechanics"* published in *"Advances in Nonlinear Variational Inequalities"*. The goal of this paper is to establish several existence results for a class of nonstandard hemivariational inequalities. Our analysis includes both the cases of bounded and unbounded closed and convex subsets in real reflexive Banach spaces. The proofs strongly rely on the KKM Principle combined with the Mosco Alternative. In the last section of the paper several applications illustrate the abstract results that were proved throughout the paper.

Throughout this paper, V will denote a real reflexive Banach space, K is a nonempty, closed and convex subset of V and (X, μ) will stand for a measure space of finite and positive measure. For a given $p > 1$ we shall denote by p' its conjugated exponent (that is $p' = p/(p-1)$) and we assume that there exists a linear and compact operator T from V into $L^p(X)$.

We are concerned with the study of a nonlinear hemivariational inequality of the type

(**P**) Find $u \in K$ such that for all $v \in K$

$$\Theta(u,v) + \int_X h(x, \bar{u}(x))\, j^0(x, \bar{u}(x); \bar{v}(x) - \bar{u}(x))\, d\mu \geq \int_X f(x, \bar{u}(x))(\bar{v}(x) - \bar{u}(x))\, d\mu,$$

where $\Theta : V \times V \to \mathbb{R}$ is a nonlinear mapping, $\bar{u}(x) := Tu(x)$ and $h, j, f : X \times \mathbb{R} \to \mathbb{R}$ are given functions. Here the notation $j^0(x, y; l)$ will stand for Clarke's generalized derivative of the mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}$ with respect to the direction $l \in \mathbb{R}$ while $\partial j(x, y)$ denotes Clarke's generalized gradient of the mapping $y \mapsto j(x, y)$ for some fixed $x \in X$.

In order to prove the existence of at least one solution of the inequality problem (**P**) we admit the following hypotheses:

($\mathbf{H_h}$) $h : X \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e. $h(\cdot, y) : X \to \mathbb{R}$ is measurable for all $y \in \mathbb{R}$, and $h(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous for a.e. $x \in X$) and there exists a constant $h_0 > 0$ such that $0 \leq h(x, y) \leq h_0$, for a.e. $x \in X$ and every $y \in \mathbb{R}$.

($\mathbf{H_f}$) $f : X \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $C_f > 0$ and $b \in L^{p'}(X)$ such that

$$|f(x, y)| \leq b(x) + C_f |y|^{p-1}$$

for a.e. $x \in X$ and all $y \in \mathbb{R}$.

($\mathbf{H_\Theta}$) $\Theta : V \times V \to \mathbb{R}$ is a nonlinear mapping which satisfies some of the bellow conditions:

($\mathbf{\Theta_1}$) $\Theta(u, u)$=0, for all $u \in V$;

($\mathbf{\Theta_2}$) the application $u \mapsto \Theta(u, v)$ is weakly upper semicontinuous for each $v \in V$, that is,

$$\limsup_{n \to \infty} \Theta(u_n, v) \leq \Theta(u, v)$$

whenever $u_n \rightharpoonup u$;

($\mathbf{\Theta_3}$) the application $v \mapsto \Theta(u, v)$ is convex for each $u \in V$;

($\mathbf{\Theta_4}$) $u \mapsto \Theta(u, v)$ is a concave;

($\mathbf{\Theta_5}$) Θ is a monotone mapping, in the sense that, $\Theta(u, v) + \Theta(v, u) \geq 0, \;\; \forall u, v \in V$.

We also assume that $j : X \times \mathbb{R} \to \mathbb{R}$ satisfies one of the following assumptions:

$(\mathbf{H_j^1})$ $j(\cdot, y) : X \to \mathbb{R}$ is measurable for all $y \in \mathbb{R}$ and there exists $k \in L^{p'}(X)$ such that

$$|j(x, y_1) - j(x, y_2)| \leq k(x)|y_1 - y_2|, \quad \forall x \in X, \ \forall y_1, y_2 \in \mathbb{R},$$

or,

$(\mathbf{H_j^2})$ $j(\cdot, y) : X \to \mathbb{R}$ is measurable for all $y \in \mathbb{R}$, the mapping $j(x, \cdot)$ is locally Lipschitz for all $x \in X$ and there exists a constant $C > 0$ such that

$$|z| \leq C(1 + |y|^{p-1}), \quad \forall x \in X, \ \forall y \in \mathbb{R}, \ \forall z \in \partial j(x, y).$$

We prove that if K is a nonempty, bounded, closed and convex subset of V and we assume that $(\mathbf{H_h})$, $(\mathbf{H_f})$, $(\mathbf{\Theta_1})$, $(\mathbf{\Theta_2})$, $(\mathbf{\Theta_3})$ and one of the conditions $(\mathbf{H_j^1})$, $(\mathbf{H_j^2})$ are fulfilled, then the inequality problem $(\mathbf{P})$ has at least one solution.

Also we prove that if K is a nonempty, bounded, closed and convex subset of V and assume that $(\mathbf{H_h})$, $(\mathbf{H_f})$, $(\mathbf{\Theta_1})$, $(\mathbf{\Theta_2})$, $(\mathbf{\Theta_4})$, $(\mathbf{\Theta_5})$ and $(\mathbf{H_j^1})$ or $(\mathbf{H_j^2})$ hold, then there exists a solution for the nonlinear hemivariational inequality $(\mathbf{P})$.

At finale we prove that if K is a nonempty, closed and convex subset of V and assume that $(\mathbf{H_h})$, $(\mathbf{H_f})$, $(\mathbf{\Theta_1})$, $(\mathbf{\Theta_2})$, $(\mathbf{\Theta_4})$, $(\mathbf{\Theta_5})$ and $(\mathbf{H_j^1})$ or $(\mathbf{H_j^2})$ hold, then there exists a solution for the nonlinear hemivariational inequality $(\mathbf{P})$, if there exists $v_0 \in K$ and $q \geq p$ such that

$$\frac{\Theta(u, v_0)}{\|u\|_V^q} \to -\infty, \quad \text{as } \|u\|_V \to \infty,$$

Chapter 9 is based on the paper *"Antiplane shear deformations of piezoelectric bodies in contact with a conductive support"* trimis spre publicare la *"Mathematiche Nachrichten"*.

In this chapter we consider a mathematical model which describes the frictional contact between a piezoelectric body and an electrically conductive support. We model the material's behavior with an electro-elastic constitutive law; the frictional contact is described with a boundary condition involving Clarke's generalized gradient and the electrical condition on the contact surface is modelled using the subdifferential of a proper, convex and lower semicontinuous function. The weak formulation of our model leads to a coupled system of a hemivariational inequality and a variational inequality. The existence of weak solutions for our model will be a direct consequence of the fact that a more general inequality, a variational-hemivaritional inequality, admits solutions. Therefore, the mathematical treatment of the model involve the theory of variational-hemivariational inequalities. The main ingredient in the proof of the existence result is a fixed point theorem for set valued mappings, due to Tarafdar. Under additional hypotheses, we prove the uniqueness of the weak solution.

The mathematical model which describes the antiplane shear deformation of a piezoelectric cylinder in frictional contact with a conductive foundation is:

Find $u,\ \varphi : \bar{\Omega} \to \mathbf{R}$ such that

$$
(\mathbf{P}) : \begin{cases}
\operatorname{div}\,(\mu(\boldsymbol{x})\nabla u(\boldsymbol{x}) + e(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})) + f_0(\boldsymbol{x}) = 0 & \text{in } \Omega,\\
\operatorname{div}\,(e(\boldsymbol{x})\nabla u(\boldsymbol{x}) - \beta(\boldsymbol{x})\nabla\varphi(\boldsymbol{x})) = q_0(\boldsymbol{x}) & \text{in } \Omega,\\
u(\boldsymbol{x}) = 0 & \text{on } \Gamma_1,\\
\varphi(\boldsymbol{x}) = 0 & \text{on } \Gamma_A,\\
\mu(\boldsymbol{x})\partial_\nu u(\boldsymbol{x}) + e(\boldsymbol{x})\partial_\nu\varphi(\boldsymbol{x}) = f_2(\boldsymbol{x}) & \text{on } \Gamma_2,\\
e(\boldsymbol{x})\partial_\nu u(\boldsymbol{x}) - \beta(\boldsymbol{x})\partial_\nu\varphi(\boldsymbol{x}) = q_B(\boldsymbol{x}) & \text{on } \Gamma_B,\\
-\mu(\boldsymbol{x})\partial_\nu u(\boldsymbol{x}) - e(\boldsymbol{x})\partial_\nu\varphi(\boldsymbol{x}) \in h(\boldsymbol{x}, u(\boldsymbol{x}))\partial j(\boldsymbol{x}, u(\boldsymbol{x})) & \text{on } \Gamma_3,\\
e(\boldsymbol{x})\partial_\nu u(\boldsymbol{x}) - \beta(\boldsymbol{x})\partial_\nu\varphi(\boldsymbol{x}) \in \partial\phi(\boldsymbol{x}, \varphi(\boldsymbol{x}) - \varphi_F(\boldsymbol{x})) & \text{on } \Gamma_3.
\end{cases}
$$

We are interested in finding weak solutions for the problem (**P**).

For this, we consider the following hypotheses:

(**H1**): $\mu \in L^\infty(\Omega)$, $\beta \in L^\infty(\Omega)$, $e \in L^\infty(\Omega)$. There exist $\beta^*, \mu^* \in \mathbf{R}$ such that $\beta(\boldsymbol{x}) \geq \beta^* > 0$ and $\mu(\boldsymbol{x}) \geq \mu^* > 0$ almost everywhere in Ω.

(**H2**): $f_0 \in L^2(\Omega)$, $q_0 \in L^2(\Omega)$, $f_2 \in L^2(\Gamma_2)$, $q_B \in L^2(\Gamma_B)$, $\varphi_F \in L^\infty(\Gamma_3)$.

(**H3**): $h : \Gamma_3 \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function (i.e. $h(\cdot, t) : \Gamma_3 \to \mathbf{R}$ is measurable, for all $t \in \mathbf{R}$, and $h(\boldsymbol{x}, \cdot) : \mathbf{R} \to \mathbf{R}$ is continuous, a.e. on Γ_3). There exists a positive constant h_0 such that $0 \leq h(\boldsymbol{x}, t) \leq h_0$, for all $t \in \mathbf{R}$, a.e. on Γ_3.

(**H4**): $j : \Gamma_3 \times \mathbf{R} \to \mathbf{R}$ is a function which is measurable with respect to the first variable, and there exists $k \in L^2(\Gamma_3)$ such that a.e. on Γ_3 and for all $t_1, t_2 \in \mathbf{R}$ we have

$$|j(\boldsymbol{x}, t_1) - j(\boldsymbol{x}, t_2)| \leq k(\boldsymbol{x})|t_1 - t_2|.$$

(**H5**): $\phi : \Gamma_3 \times \mathbf{R} \to \mathbf{R}$ is a functional such that $\phi(\cdot, t) : \Gamma_3 \to \mathbf{R}$ is measurable for each $t \in \mathbf{R}$ and $\phi(\boldsymbol{x}, \cdot) : \mathbf{R} \to \mathbf{R}$ is convex and lower semicontinuous a.e. on Γ_3.

We prove that if conditions (**H1**)-(**H5**) are fulfilled, then there exists at least one weak solution for problem (**P**).

We note that, under the the hypotheses (**H1**)-(**H5**), the uniqueness of the weak solution of Problem (**P**) is an *open problem*.

Let us assume the following hypotheses:

(**H6**) there exists $m > 0$ such that $(\eta_1 - \eta_2)(t_1 - t_2) \geq -m|t_1 - t_2|^2$, for all $t_1, t_2 \in \mathbf{R}$, all $\eta_i \in h(\boldsymbol{x}, t_i)\partial j(\boldsymbol{x}, t_i)$, $i \in \{1, 2\}$, and a.e. on Γ_3;

(**H7**) $\min\{\mu^*, \beta^*\} > mC^2$,

where $C > 0$ appears in the inequality $\|v\|_{L^2(\Gamma_3)} \leq C\|v\|_V$ for all $v \in V$.

We prove that if conditions (**H1**)-(**H7**) are fulfilled, then problem (**P**) has a unique weak solution.

The nine chapters presented above are followed by a rich bibliography, containing 184 reference papers.