**Resumé of the Ph.D Thesis**

**"Topological Methods in the Study of Boundary Value Problems"**,

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In this thesis we are concerned with some important topics in the theory of partial differential equations. All the problems studied here are containing nonlinearities and we are concerned with the existence of solutions (sometimes nonexistence, sometimes uniqueness or multiplicity). We incorporate the principle of the symmetry into the overall organization of the text, which is subdivided into three parts. This partition of our paper is given by the class of the problems studied, as follows:

- Part I (Chapters 2-5): elliptic problems with large solutions;
- Part II (Chapters 6-9): variational problems with $p(x)$-growth conditions;
- Part III (Chapters 10-12): degenerate and singular boundary value problems.

The first chapter of every part (we refer to Chapters 2, 6 and 10) contains some preliminaries, disposed into three sections. The first section is an introductory section where we present some history notes, underlining the pioneering papers for each field. The second section contains some significant recent results and the third section gives the physical motivation of the study.

With the exception of Chapter 1, which was reserved to the Introduction, the other eight chapters (Chapters 3-5, 7-9 and 11-12) are based on published articles and on articles accepted to be published.

Chapter 3 is based on the paper "Entire large solutions for logistic-type equations", which will appear in *Annals of the University of Craiova*. In this chapter we consider the following class of semilinear elliptic equations

$$\begin{cases} \Delta u = u + e^{-|x|^a} u^\alpha f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \geq 3$, $a \geq 1$, $\alpha > 1$ and $f$ is under the assumptions

$$f \in C^1([0, \ \infty)), \quad f' \geq 0, \quad f \geq 1. \tag{2}$$

Firstly we prove that the equation

$$\begin{cases} \Delta u = e^{-|x|^a} u^\alpha f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N \end{cases}$$

has a positive entire large solution. Then we discuss the particular case of equation (1) when $a = 1$ and $\alpha > 2$, namely

$$\begin{cases} \Delta u = u + e^{-|x|}u^\alpha f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{3}$$

The main result asserts the fact that even though both equations

$$\begin{cases} \Delta u = u & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases}$$

and

$$\begin{cases} \Delta u = e^{-|x|}u^\alpha f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases}$$

have positive entire large solutions, equation (3) has no such solutions. The proof is based on the classical method of reduction to absurdity. In our calculus we rely on the properties of the special functions gamma and Bessel, and we use the maximum principle. It remains to be seen whether or not problem (1) admits positive entire large solutions. Although we bring some arguments supporting the idea that for $a$ sufficiently large the answer is positive, we let this matter as an open problem.

Chapter 4 is based on the paper "On the existence and nonexistence of positive entire large solutions for semilinear elliptic equations", which will appear in the journal *Analele Ştiinţifice ale Universităţii Ovidius Constanţa*. Here we improve the results obtained in Chapter 3 by discussing the following class of semilinear elliptic equations

$$\begin{cases} \Delta u = p_1(x)u^\alpha + p_2(x)u^\beta f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4}$$

where $N \geq 3$ and $f$ is under the assumptions (2). In this chapter we will not give only existence results, but some nonexistence results, too. Being under the assumptions that $\alpha, \beta > 1$ and $p_1, p_2 \in C^{0,\mu}_{\text{loc}}(\mathbb{R}^N)$ ($N \geq 3$, $0 < \mu < 1$) are $c$-positive in $\Omega_n$ (i.e. for every $x_0 \in \Omega_n$ with $p_1(x_0) = 0$ (respectively $p_2(x_0) = 0$) there is a domain $\Omega_0 \ni x_0$ such that $\overline{\Omega}_0 \subset \Omega_n$ and $p_1 > 0$ (respectively $p_2 > 0$) on $\partial\Omega_0$), where by $\Omega_n$ we understand the ball $|x| < n$, if problem

$$\begin{cases} \Delta u = p_1(x)u^\alpha & \text{in } \mathbb{R}^N, \\ u \geq 0,\ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{5}$$

or problem

$$\begin{cases} \Delta u = p_2(x)u^\beta f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{6}$$

has no positive entire large solutions, nor does problem (4).

But what happens when both problems (5) and (6) have positive entire large solutions? In Section 4.3 we give some examples in both directions. Moreover, if we assume $N \geq 3$, $\beta > 2$, $f$ verifies (2), $p_1 \in C(\mathbb{R}^N)$ satisfies $p_1(x) = p_1(|x|) \geq 1$ with

$$\int_0^\infty r p_1(r) dr = \infty,$$

and $p_2 \in C^{0,\mu}_{\text{loc}}(\mathbb{R}^N)$ $(0 < \mu < 1)$ satisfies $p_2(x) \geq e^{-|x|}$ with

$$\int_0^\infty r M_{p_2}(r) dr < \infty,$$

where $M_{p_2}(r) \equiv \max_{|x|=r} p_2(x)$ then we prove that the particular case of problem (4),

$$\begin{cases} \Delta u = p_1(|x|)u + p_2(x)u^\beta f(u) & \text{in } \mathbb{R}^N, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases}$$

has no positive entire large solutions, while both problems (6) and the particular case of problem (5),

$$\begin{cases} \Delta u = p_1(|x|)u & \text{in } \mathbb{R}^N, \\ u \geq 0, \ u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases}$$

have positive entire large solutions.

Chapter 5 is based on the paper "Uniqueness of singular radial solutions for a class of quasi-linear problems" which will appear in the *Bulletin of the Belgian Mathematical Society – Simon Stevin*. In this chapter we establish the uniqueness and the blow-up rates of the singular value problem:

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} - b(x)u^q & \text{in } B_R(x_0), \\ u > 0 & \text{in } B_R(x_0), \\ u = \infty & \text{on } \partial B_R(x_0), \end{cases} \tag{7}$$

where $B_R(x_0)$ is the ball of radius $R$ centered at $x_0 \in \mathbb{R}^N$, $N \geq 3$, $\lambda > 0$, $b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, $b > 0$, $q > p - 1 > 1$ and we denoted by $\Delta_p$ the $p$-Laplace operator given by

$$\Delta_p u = \operatorname{div}\big(|\nabla u(x)|^{p-2}\nabla u(x)\big).$$

The boundary condition in (7) is understood as $u(x) \to +\infty$ when $d(x) = \operatorname{dist}(x, \partial\Omega) \to 0^+$.

The main result of this chapter is based on some auxiliary theorems in which we are extending some already known results from the case $p = 2$ to the case $p > 2$. We prove that there exists a unique large solution $u(x)$ to the problem (7) which satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{K(b^*(\|x - x_0\|))^{-\beta}} = 1,$$

where $\lambda > 0$, the potential function $b$ is a positive radially symmetric function, $d(x) = \operatorname{dist}(x, \partial B_R(x_0))$, $K$ is a constant defined by

$$K = \left[(p-1)[(\beta+1)C_0 - 1]\beta^{p-1}(C_0 b_0)^{(p-2)/2}\right]^{\frac{1}{q-p+1}},$$

with

$$\beta = \frac{p}{2(q-p+1)},\ q > p - 1 > 1,\ b_0 = b(R) > 0,\ C_0 = \lim_{r\to R} \frac{(B(r))^2}{b^*(r)b(r)} \geq 1$$

and

$$B(r) = \int_r^R b(s)ds,\ b^*(r) = \int_r^R \int_s^R b(t)dtds.$$

Chapters 7-9 are dedicated to the study of the problems with $p(x)$-growth conditions. In Chapters 7 and 8 we provide some existence results using as main argument the mountain-pass theorem. One of the main ideas in searching weak solutions for PDEs is based on the critical point theory. More precisely, an equation can be associated with an energetic functional whose critical points will offer the solutions of the equation.

In these chapters we present some results on the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, since the $p(x)$-growth conditions can be regarded as a very important class of nonstandard growth conditions and the interest comes from the applicability in elastic mechanics and mathematical modelling of non-Newtonian fluids. Most materials can be modeled with sufficient accuracy using classical Lebesgue and Sobolev spaces, $L^p$ and $W^{1,p}$, where p is a fixed constant. For some materials with inhomogeneities, like electrorheological fluids, this is not adequate, but rather the exponent p should be able to vary.

Chapter 7 is based on the paper "Existence of nontrivial weak solutions for a class of problems with $p(x)$-growth conditions" published in *Proceedings of the National Session of Student Scientific Communications, Iaşi 2007.* Using as our main tool the mountain-pass theorem of Ambrosetti and Rabinowitz, we establish the existence of weak solutions for the following problem:

$$\begin{cases} -\Delta_{p(x)}u(x) + a(x)|u(x)|^{p(x)-2}u = |u(x)|^{q(x)-1}u & \text{in } \Omega, \\ u \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $p, q : \overline{\Omega} \to \mathbb{R}$ are continuous functions with $2 \leq \min_{\overline{\Omega}} p(x) < \max_{\overline{\Omega}} p(x) < N$, $\max_{\overline{\Omega}} p(x) < \min_{\overline{\Omega}} q(x) + 1$, $q(x) \leq N$, $q(x) + 1 < Np(x)/(N - p(x))$ for all $x \in \overline{\Omega}$ and $a : \overline{\Omega} \to \mathbb{R}$ satisfies the condition: $a \in L^\infty(\overline{\Omega})$ and there exists $a_0 > 0$ such that $a(x) \geq a_0$, for any $x \in \Omega$.

We denoted by $\Delta_{p(x)}$ the $p(x)$-Laplace operator, i.e.

$$\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u(x)|^{p(x)-2} \nabla u(x)\right).$$

The results of Chapter 8 are based on the following articles: "Existence of solutions for an elliptic equation involving the $p(x)$-Laplace operator", published in *Electronic Journal of Differential Equations* and "Fraternization in ... Mathematics", published in *Proceedings of the International Conference of Young Scientists, affiliated to CASC, Kisinev 2006.* In Chapter 8 we also use the mountain-pass theorem of Ambrosetti and Rabinowitz to prove the existence of solutions for the problem

$$\begin{cases} -\Delta_{p(x)} u(x) + b(x)|u(x)|^{p(x)-2} u = f(x, u), & \text{for} \quad x \in \mathbb{R}^N \\ u \in W_0^{1,p(x)}(\mathbb{R}^N) \end{cases}$$

where $N \geq 3$, $p : \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous with $2 \leq \operatorname{ess\,inf}_{\mathbb{R}^N} p(x) < \operatorname{ess\,sup}_{\mathbb{R}^N} p(x) < N$, $b : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are two functions which satisfy the hypotheses:

(b1) $b \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and there exists $b_0 > 0$ such that $b(x) \geq b_0$, for any $x \in \mathbb{R}^N$;

(f1) $f \in C^1(\mathbb{R}^N \times \mathbb{R})$, with $f = f(x, z)$, $f(x, 0) = 0$ and $\lim\limits_{z \to 0} \frac{f_z(x,z)}{|z|^{p^+-2}} = 0$, for all $x \in \mathbb{R}^N$;

(f2) $p^+ < \frac{Np^-}{N-p^-}$ and there exist $s \in (p^+ - 1, Np^-/(N - p^-) - 1)$, $\theta \in (s, Np^-/(N - p^-))$ and $g_1 \in L^\infty(\mathbb{R}^N) \cap L^{\theta/(\theta - p^+ + 1)}(\mathbb{R}^N)$, $g_2 \in L^\infty(\mathbb{R}^N) \cap L^{\theta/(\theta - s)}(\mathbb{R}^N)$, with $g_1(x), g_2(x) \geq 0$ such that

$$|f_z(x, z)| \leq g_1(x)|z|^{p^+-2} + g_2(x)|z|^{s-1}, \quad \forall\, x \in \mathbb{R}^N,\ \forall\, z \in \mathbb{R}.$$

(f3) there exists $\mu > p^+$ such that

$$0 < \mu F(x, z) = \mu \int_0^z f(x, t)dt \leq z f(x, z), \quad \forall\, x \in \mathbb{R}^N, \forall\, z \in \mathbb{R} \setminus \{0\}.$$

Chapter 9 is based on the paper "Existence and multiplicity of solutions for a Neumann problem involving variable exponent growth conditions" published in *Glasgow Mathematical Journal.* We discuss the existence of solutions for the Neumann problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = f(u), & \text{for} \quad x \in \Omega \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{for} \quad x \in \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $p \in C(\overline{\Omega})$ with $1 < p(x) < N$ for all $x \in \overline{\Omega}$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function given by formula

$$f(t) = \begin{cases} |t|^{a-1}t, & \text{for} \quad |t| \leq \left(\dfrac{1}{2}\right)^{\frac{1}{a-1}} \\ t - |t|^{a-1}t, & \text{for} \quad |t| > \left(\dfrac{1}{2}\right)^{\frac{1}{a-1}} \end{cases},$$

where $a$ is a positive real number. Our main result asserts the existence of two solutions if $p^+ < a < \frac{Np^-}{N-p^-}$, using as main tool an abstract linking argument due to Brézis and Nirenberg.

In Chapters 11-12 we focus on the class of problems with degeneracies and singularities. Here we justify the existence and uniqueness of weak solutions for some problems modelling the antiplane shear deformation of a cylindrical body. In order to do that we introduce some weighted spaces and we establish some new results. From the mathematical point of view, the novelty comes from the fact that we consider problems where the function $u$ vanishes only on a part (of positive Lebesgue measure) of the boundary, instead of vanishing on the whole boundary. From the mechanical point of view, the main novelty of our study comes from the fact that, unlike the research already made in this field, we are able to consider the degenerate situation when $\inf_{\boldsymbol{x} \in \overline{\Omega}} \mu(\boldsymbol{x}) = 0$. The results obtained here are based on the work "Weak solutions for antiplane models involving elastic materials with degeneracies" which will appear in *Zeitschrift für Angewandte Mathematik und Physik (ZAMP).*

In Chapter 11 we consider the following class of boundary value problems

$$\begin{aligned} \operatorname{div}\big(\mu^2(\boldsymbol{x})\nabla u(\boldsymbol{x})\big) + f_0(\boldsymbol{x}) &= 0 && \text{in } \Omega, \\ u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_1, \\ \mu^2(\boldsymbol{x})\frac{\partial u}{\partial \nu}(\boldsymbol{x}) &= f_2(\boldsymbol{x}) && \text{on } \Gamma_2, \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$ is an open, bounded, connected subset, with Lipschitz continuous boundary $\Gamma$ partitioned in two measurable parts $\Gamma_1$, $\Gamma_2$, such that the Lebesgue measure of $\Gamma_1$ is strictly positive. Under the assumptions

$$\mu \in L^2(\Omega), \quad \mu^{-1} \in L^2(\Omega), \quad \mu(\boldsymbol{x}) \neq 0 \text{ a.e. on } \Omega, \tag{8}$$

$$\inf_{\boldsymbol{x} \in \overline{\Omega}} \mu^2(\boldsymbol{x}) = 0, \quad \sup_{\boldsymbol{x} \in \overline{\Omega}} \mu^2(\boldsymbol{x}) = \infty \tag{9}$$

and

$$f_0 \in L^2(\Omega), \quad f_2 \in L^\infty(\Gamma_2), \tag{10}$$

we provide the existence and the uniqueness of a weak solution using the Lax-Milgram theorem.

In Chapter 12 we study the problem

$$\begin{cases} \operatorname{div}(a(\boldsymbol{x}, \nabla u(\boldsymbol{x}))) + f_0(\boldsymbol{x}) = 0 & \text{in } \Omega, \\ u(\boldsymbol{x}) = 0 & \text{on } \Gamma_1, \\ a(\boldsymbol{x}, \nabla u(\boldsymbol{x})) \cdot \boldsymbol{\nu}(\boldsymbol{x}) = f_2(\boldsymbol{x}) & \text{on } \Gamma_2, \\ -a(\boldsymbol{x}, \nabla u(\boldsymbol{x})) \cdot \boldsymbol{\nu}(\boldsymbol{x}) \in \partial\varphi_x(u(\boldsymbol{x})) & \text{on } \Gamma_3, \end{cases}$$

where $\Omega$ is a regular domain, $a : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$, $a(\boldsymbol{x}, \boldsymbol{v}) = [\mu^2(\boldsymbol{x}) + \beta(\boldsymbol{x})]\boldsymbol{v} - 2\beta(\boldsymbol{x})P_{\widetilde{K}}\frac{1}{2}\boldsymbol{v}$, $P_{\widetilde{K}}$ is the projection operator on $\widetilde{K} = \overline{B_k(O_{\mathbb{R}^2})}$, $\varphi_x : \mathbb{R} \to \mathbb{R}$, $\varphi_x(s) = g(\boldsymbol{x})|s|$ and $g \in L^\infty(\Gamma_3)$, $g \geq 0$ a.e. on $\Gamma_3$. Also, to the hypotheses (8), (9) and (10) we add $\beta \in L^\infty(\Omega)$ and there exists $c > 0$ such that $0 < \beta(\boldsymbol{x}) \leq c\,\mu_0(\boldsymbol{x})$ a.e. in $\Omega$. Then we establish the existence of a unique weak solution by using a result from the theory of variational inequalities.

The twelve chapters presented above are followed by a rich bibliography, containing 227 reference papers.