# THE CALCULATION OF INTEGRALS AND SERIES WITH HIGH ACCURACY

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- summary of the doctoral thesis -

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## Contents



## 1. Introduction

The present paper, concerning the title, refers to the study of integrals and series, being at the border between the mathematical analysis and informatics. The integrals and the series are used in various domain of science and technique, in mechanics, physics, chemistry, engineering, aeronautics, astronomy, economy, biology, etc. For example, the elliptical integrals are used for the dynamics of a material point subjected to connections, in supersonic aerodynamics, for the expression of the lift factor for of a thin delta wing with subsonic edges of attack; the logarithmic integrals $\int_0^1 p(x) \cdot \log(\log(1/x))\, dx$, with $p(x)$ as rational function (see [1], [2]), are used in the statistical physics and in the lattice theory; the abelian integrals are used in the differential geometry; the integrals of the form $\int_0^1 x^r (1 \pm x)^{-1} \cdot \log^k (1 \pm x) \cdot \log^l x dx$, [3] have values established depending on the values of zeta function, a function used in quantical physics. *The BBP-Ramanujan,* type series, hypergeometric, eulerian, various harmonic subseries are used for the calculation of some constants with high accuracy, in different basis of enumeration, for example the number $\pi$, which squeezed the minds of many mathematicians since the beginning of times, or the number $\exp(1)$, or the values of the zeta function, in its various forms, [3], etc.

## 2. Integer relation and the PSLQ algorithm

In this chapter I defined the term of integer relation of a given vector and I discussed the evolution of various algorithms (the Euclid algorithm, LLL, HJLS, PSLQ, Multi-Pair) which can be obtained. I adapted the algorithms for the *Maple 7* language in order to be used. Their source programme is to be found in Annex B, together with some of their applications. At the end of the chapter there are indicated several alternative software where can found implementations of these algorithms.

We note $\mathbb{K}$ the real, complex or quaternion numbers and with $\mathbb{O}(\mathbb{K})$ the ordinary integer numbers, Gauss integer numbers or Hamilton integer numbers.

**Definition 1.** *A vector $x = (x_1, x_2, ..., x_n)$ from $\mathbb{K}^n$ we say that it is in* integer relation *if there is a vector $m = (m_1, m_2, ...m_n) \in \mathbb{O}(\mathbb{K})^n$, $m \neq 0$, so that*

$$m_1x_1 + m_2x_2 + ... + m_nx_n = 0. \tag{1}$$

*The vector $m$ different from zero is called* integer relation *for $x$.*

**Examples:**
**1)** $x = \left(2/3,\ 1/2,\ 5/6\right)$, $m = \left(1,\ 2,\ -2\right)$;
**2)** $x = \left(\alpha,\ \pi,\ e,\ 1\right)$, $m = \left(-72,\ 54,\ -63,\ 8\right)$, where $\alpha = -3\pi/4 - 7e/8 + 1/9$.

## 3. Some applications of the integer relation

The chapter starts with the justification for using the various algorithms in determining the integer relation, as well as accompanying them with rigorous demonstrations. The relations, discovered with the computer aid can be true up to a number of decimals, that it is imposed their rigorous demonstration. Though approximate relations can be used in other domains of the science (in physics, economy, geography, in various statistical analysis of trend), where a sufficient number of decimals is enough. An eloquent example is the use of the number $\pi$, which in antiquity, depending on the needs, it was approximated as an integer and then, once the need for precision increased, and as the society developed, it started to be used with more and more precise decimals. The first calculation algorithm for this number was given by *Arhimede din Siracusa* (287-212 b. C.) who used *the exhaustive method of Eudoxiu* $(3 + 10/71 < \pi < 3 + 1/7)$.

There are analyzed several applications of the integer relation: *the recognition of certain numerical constants, finding the minimal polynom* and *obtaining the identities.* With the recognition of numerical constants dealt various mathematicians and computer scientists performing software. Such software is Inverse Symbolic Calculator (ISC) which can be found at : *http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html.*

In the present paper I used a software package performed by Alain Meichsner in a Maple version, which I updated it for the version *Maple 7,* and which offers a lot of facilities. The package contains three lists of constants $B_1$, $B_2$, $B_3$, which are being modified depending on the needs of the problems we are solving. Thus, I found the values and then I demonstrated for several logarithmic integrals, an exponential integral, a class of integrals and one of trigonometric series which contain the function $\operatorname{sinc}(x) = x^{-1}\sin(x)$. There are being analyzed two double integrals, proposed by the *American*

*Mathematical Monthly* Magazine, the problems number 11275 'and 11277 from 2007, in the demonstration of which there are particular cases of some generalized integrals in chapter 4. Also for the "Recognition of numerical constants" there are studied the binomial series of Lehmer and the ones of Apéry type. In 1979, Roger Apéry used such a series to demonstrate the irrationality of the constant $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$, and as a result it was called the *Apéry constant.* Finding the minimal polynom is an immediate application of the LLL algorithm, being useful in algebra. We choose two significant examples which we accompanied with demonstrations.

The subchapter "Obtaining identities" has as examples two types: *Machin type identities* and *ladder type identities* , used in chapter 5, in order to obtain sums of BBP-Ramanujan type. The Machin type identities are integer linear combinations of values of the arctangent function. The name comes from the one who obtained for the first time such an identity, in 1706, for a quick calculation of the $\pi$ number and using more decimals. We realized a few software in Maple 7 in order to trace them and we obtained lists of such identities. Their demonstrations is elementary. The ladder type identities are integer linear combinations of values of the polylogarithm function $\mathrm{Li}_n(z) = \sum_{r=1}^{\infty} z^r/r^n$, with $|z| \leq 1$, and produced by values of zeta functions and exponents of the natural logarithmic function. Such types of identities have been discovered for the first time by L. Euler in 1768 and by J. Landen in 1780, and then the theoretical study was done by L. Lewin in 1991. With the help of PSLQ algorithm I found lot of such expressions, and they can be found in Annex D.

Henceforth are exemplified two identities with series: *the BBP series* and *the Bill Gosper formulae.* The first one was discovered by experimental methods, with the help of PSLQ algorithm, by David Bailey, Peter Borwein and Simion Plouffe in 1995 . The value of this series is given by the fact that it helps at the calculation of $\pi$ number, in hexadecimal basis, starting from a given hexadecimal, and there is no need to know the precedent ones. In Maple, I realized a software for the calculation of a great number of hexadecimals for $\pi$ number, based on this series. It can be found in Annex C. After discovering the BBP series, there appeared other of this type in order to find some methods of calculation for other constants, using as many as possible exact decimals (see chapter 5). The Bill Gosper formulae is a binomial series which can be used for the calculation of $\pi$. A demonstration of this formula uses the Beta function $\beta(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx$ and values of the polylogarithmic series transforming the series into a rational integral.

## 4. The calculation of some integrals

The chapters 4 and 5 are the center of this paper, the first one refers to study of integrals and the second one is reserved for the series. The chapter 4 encloses three important subchapters: *the integration of elementary functions*, *an application of Dirichlet L series* and *the use of multiple zeta function.*

### 4.1. The integration of elementary functions

This subchapter is an introduction which refers to the calculation of integrals where are mentioned the results obtained by Niels H. Abel, Joseph Liouville, D. Mordukhai-Boltovskoi and Joseph-Fels Ritt, as well as cases where certain integrals are elementary (for example, Cebâsev integral), or they are not elementary (for example, elliptical integrals or some exponential or logarithmic integrals). In the following two subchapters I studied types of logarithmic integrals which values depend on the values of certain hypertranscedental functions: gamma function (hipertranscedentality demonstrated by O. Hölder, in 1887) and zeta function (D. Hilbert, in 1900, demonstrated the hypertranscedentality of the Riemann zeta function).

### 4.2. An application of Dirichlet L series

For the integrals of the following type

$$I_{-q} = \int_0^1 \frac{\sum_{n=1}^{q-1} \chi_{-q}(n)\, x^{n-1}}{1-x^q} \ln\left(\ln\frac{1}{x}\right) dx, \tag{2}$$

where $\chi_{-q}$ is Dirichlet character (mod $q$), which I presented at the $6^{th}$ Congress of Romanian Mathematicians in Bucharest held on the $28^{th}$ June - $4^{th}$ July, 2007 [1], as well as in [2], there can be found with the help of the computer various expressions in the values of the gamma. For example,

$$\begin{aligned}\frac{\sqrt{3}}{\pi} I_{-15;-3} &= \int_0^1 \frac{\sqrt{3}\sum_{n=1}^{14} \chi_{-15;-3}(n)\, x^{n-1}}{\pi\left(1-x^{15}\right)} \ln\ln\frac{1}{x} dx \\ &= \int_0^1 \frac{\sqrt{3}\left(x^8-x^7+x^3-x+1\right)}{\pi\left(x^2+x+1\right)\left(x^8-x^7+x^5-x^4+x^3-x+1\right)} \ln\ln\frac{1}{x} dx \\ &= \frac{1}{5}\ln\left(\frac{3^3\Gamma^{12}(2/3)}{2^4\pi^4\sqrt[3]{5}}\right).\end{aligned}$$

For these types of integrals I demonstrated two theorems which justify these formulae:

**Theorem 2.** *If* $\chi_{-\Delta}$ *is an* odd primitive character $(mod\,\Delta)$ *then :*

$$I_{-\Delta} = \int_0^1 \frac{\sum\limits_{n=1}^{\Delta-1} \chi_{-\Delta}(n)\, x^{n-1}}{1-x^{\Delta}} \ln\ln\frac{1}{x} dx = \begin{cases} \frac{\pi}{\sqrt{3}} \ln\left(\frac{\sqrt{3}\cdot\Gamma^2(2/3)}{(2\pi)^{2/3}}\right) & if\ \ \Delta = 3 \\ \pi\ln\left(\frac{\Gamma(3/4)}{\sqrt[4]{\pi}}\right) & if\ \ \Delta = 4 \\ \frac{\pi}{\sqrt{\Delta}}\left\{\ln 2\pi - \sum\limits_{r=1}^{\Delta-1} \chi_{-\Delta}(r) \ln\Gamma\left(\frac{r}{\Delta}\right)\right\} & if\ \ \Delta > 4 \end{cases}. \tag{3}$$

**Theorem 3.** *If* $\chi_{-q;-\Delta}$ *is an* odd character *(mod q) induced by an* odd primitive character $\chi_{-\Delta}$ $(mod\,\Delta)$ *then :*

$$(4)\quad I_{-q;-\Delta}=\int_0^1\frac{\sum\limits_{n=1}^{q-1}\chi_{-q;-\Delta}(n)\,x^{n-1}}{1-x^q}\ln\ln\frac{1}{x}dx$$
$$=\left(I_{-\Delta}+L(1,\chi_{-\Delta})\sum_{\substack{p/q\\ p\ prime}}^{q-1}\frac{\chi_{-\Delta}(p)\ln p}{p-\chi_{-\Delta}(p)}\right)\prod_{\substack{p/q\\ p\ prime}}\left(1-\frac{\chi_{-\Delta}(p)}{p}\right),$$

*where*

$$(5)\qquad L(1,\chi_{-\Delta})=\begin{cases}\pi/\left(3\sqrt{3}\right) & if\ \ \Delta=3\\ \pi/4 & if\ \ \Delta=4\\ \frac{\pi}{-\Delta\sqrt{\Delta}}\sum\limits_{n=1}^{\Delta-1}n\cdot\chi_{-\Delta}(n) & if\ \ \Delta>4\end{cases}.$$

As we can see in the expression of the function from which we calculate in the integral there are found the values of the character $\chi$ of Dirichlet. That is way it is imposed to know the values for this function. We built a software named *Lag*, in Maple 7, which helped us find these values that we grouped in several tables in annex A from this paper. The tables contain the values of the primitive modulo characters $|\Delta|\leq 40$ and the ones for the modulo inducted odd character $|q|\leq 40$, by the primitive modulo characters $|\Delta|\leq 40$ . The method used to find the formulae, using the computer, is based on the package presented in subchapter "Recognition of the numerical constants" from chapter 3. Thus, we obtained a list of integrals which can be extended at any moment. Similar integrals where studied by V. Adamchik, with applications *problems of statistical physics* and in *the theory of lattices*, as well as Baxter, Temperley and Ashley, in *problems for coloring the graphs*. Adamchik discovered formulae which contain the values of the digamma function $\psi(s)=\Gamma'(s)/\Gamma(s)$ and for the partial derivate of Hurwitz zeta function $\zeta^{(1,0)}(s,z)=\partial\zeta(s,z)/\partial s$, where $\zeta(s,z)=\sum_{n=0}^{\infty}(n+z)^{-s}$, with $\operatorname{Re}s>1$, $0<z\leq 1$. Medina and Moll discovered other formulae which I arranged more structured . These are expressions which contain the polylogarithmic function $\operatorname{Li}(s,x)=\sum_{n=1}^{\infty}x^n n^{-s}$, $|x|<1$ and its partial derivate $\operatorname{Li}^{(1,0)}(s,x)=\partial\operatorname{Li}(s,x)/\partial s$.

### 4.3. Using the zeta multiple function for the calculation of some integrals

In this last subchapter I studied integrals of the following type

$$(6)\qquad I^{\pm}_{k,l,r,m}=\int_0^1\frac{\left(-\log(1\pm x)\right)^k}{(1\pm x)^m}\cdot x^r\left(-\log x\right)^l dx,$$
$$(7)\qquad J^{\pm}_{l,r,m,k}=\int_0^1\frac{x^{kr+k-1}}{\left(1\pm x^k\right)^m}\cdot\left(-\log x\right)^l dx,$$

where $k$, $l$, $r$, $m$ are integers, obtaining expressions which contain values for the multiple zeta function of Euler-Zagier and for extended multiple zeta function (see [3], [5]):

**Definition 4.** *We call* multiple $\zeta$ function a of Euler-Zagier *the complex function :*

$$(8) \qquad \zeta_l(s_1, s_2, ..., s_l) = \sum_{n_1>n_2>n_3>...>n_l\geq 1} \frac{1}{n_1^{s_1}\cdot n_2^{s_2}...\cdot n_l^{s_l}};$$

*where the addition is made after all integer values* $n_1 > n_2 > n_3 > ... > n_l$, *with* $l \in \mathbb{N}^*$.

**Definition 5.** *We call* multiple extended $\tilde{\zeta}$ function *the complex function*:

$$(9) \quad \tilde{\zeta}_l(s_1, s_2, ..., s_l; \sigma_1, \sigma_2, ..., \sigma_l) = \sum_{n_1>n_2>n_3>...>n_l\geq 1} \frac{\sigma_1^{n_1}\cdot\sigma_2^{n_2}\cdot...\cdot\sigma_l^{n_l}}{n_1^{s_1}\cdot n_2^{s_2}\cdot...\cdot n_l^{s_l}}, \text{ where } \sigma_i = \pm 1.$$

*In particular*:

$$(10) \qquad \tilde{\zeta}_l(s_1, s_2, ..., s_l) = \sum_{n_1>n_2>n_3>...>n_l\geq 1} \frac{\sigma_1^{n_1}\cdot\sigma_2^{n_2}\cdot...\cdot\sigma_l^{n_l}}{n_1^{|s_1|}\cdot n_2^{|s_2|}\cdot...\cdot n_l^{|s_l|}},$$

*where* $l \in \mathbb{N}^*, s_1, s_2, ..., s_l \in \mathbb{Z}^*, \sigma_j = signum(s_j)$.

Particular cases of these expressions were known by L. Euler, as they contained only values of the zeta function in a single variable. They can be found in I. S. Gradzhteyn and I. M. Ryzhyk book , for example

$$(11) \qquad \int_0^1 \frac{1}{1+x}\cdot\log x dx = \tilde{\zeta}_1(-2) = -\frac{\pi^2}{12}, \qquad \text{(formula 4.231.1)}$$

$$(12) \qquad \int_0^1 \frac{1}{1-x}\cdot\log x dx = -\zeta_1(2) = -\frac{\pi^2}{6}, \qquad \text{(formula 4.231.2)}$$

The Euler-Zagier sums, which appear in the definitions of zeta functions, occupy an important role in the theory of nodes and in the theory of quantic fields. The formulae connected to these integrals can be found with the help of PSLQ algorithm or with the package for identifying the constants realized in Maple. The demonstrations are given in the form of theorems or corollaries:

**Theorem 6.** *If* $k, r$ *and* $l$ *are nonnegative integers,* $k, l, r \geq 0$, $l \geq 1.$*,then:*

$$I^{\pm}_{k,l,r,1} = \int_0^1 \frac{(-\log(1\pm x))^k}{1\pm x}\cdot x^r(-\log x)^l dx$$
$$= \mp k!l! \sum_{n_1>n_2>n_3>...>n_{k+1}\geq 1} \frac{(\mp 1)^{n_1}}{(n_1+r)^{l+1}\cdot n_2\cdot n_3\cdot...\cdot n_{k+1}}.$$

**Corollary 7.** *Suppose that* $k$ *and* $l$ *are integers with,* $k \geq 0$, $l \geq 1$. *Then :*

$$I^{\pm}_{k,l,0,1} = \int_0^1 \frac{(-\log(1\pm x))^k}{1\pm x}\cdot(-\log x)^l dx = \mp k!l!\tilde{\zeta}_{k+1}(\mp(l+1), \{1\}_k);$$

*and*

$$I^{+}_{k,0,0,1} = \int_0^1 \frac{(-\log(1+x))^k}{1+x} dx = -k!\tilde{\zeta}_{k+1}(-1, \{1\}_k).$$

**Corollary 8.** *Suppose that $k$ and $l$ are integers with, $k \geq 0$, $l \geq 1$. Then :*

$$I^{\pm}_{k,l,1,1} = \int_0^1 \frac{(-\log(1 \pm x))^k}{1 \pm x} \cdot x(-\log x)^l dx$$
$$= k!l! \left( \tilde{\zeta}_{k+1}\left(\mp(l+1), \{1\}_k\right) \pm \sum_{n_1=n_2>n_3>\ldots>n_{k+1}\geq 1} \frac{(\mp 1)^{n_1}}{(n_2+1)^{l+1} \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_{k+1}} \right),$$

*and*

$$I^{+}_{k,0,1,1} = \int_0^1 \frac{(-\log(1 + x))^k}{1 + x} \cdot x dx$$
$$= k! \left( \tilde{\zeta}_{k+1}\left(-1, \{1\}_k\right) + \sum_{n_1=n_2>n_3>\ldots>n_{k+1}\geq 1} \frac{(-1)^{n_1}}{(n_2+1) \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_{k+1}} \right).$$

*In particular*

$$I^{\pm}_{0,l,1,1} = \int_0^1 \frac{1}{1 \pm x} \cdot x(-\log x)^l dx = l!\left(\tilde{\zeta}\left(\mp(l+1)\right) \pm 1\right), \quad \text{where } l \in \mathbb{N}^*$$

$$I^{+}_{0,0,1,1} = \int_0^1 \frac{1}{1 + x} \cdot x dx = \tilde{\zeta}(-1) + 1 = 1 - \log(2),$$

$$I^{-}_{1,l,1,1} = -\int_0^1 \frac{\log(1 - x)}{1 - x} \cdot x(-\log x)^l dx$$
$$= -l!\left(\zeta_2(l+1, 1) - l - 1 + \sum\nolimits_{i=2}^{l+1} \zeta(i)\right), \text{ where } l \in \mathbb{N}^*$$

$$I^{+}_{1,l,1,1} = -\int_0^1 \frac{\log(1 + x)}{1 + x} \cdot x(-\log x)^l dx$$
$$= -l!\left(\tilde{\zeta}_2(-l-1, 1) + l + 1 + 2\tilde{\zeta}(-1) + \sum\nolimits_{i=2}^{l+1} \tilde{\zeta}(-i)\right), \text{ where } l \in \mathbb{N}$$

**Corollary 9.** *Let $l$ and $r$ be integer such that, $l \geq 1$, $r \geq 0$. Then:*

$$(13) \qquad I^{\pm}_{0,l,r,1} = \int_0^1 \frac{x^r}{1 \pm x} \cdot (-\log x)^l dx = l!(\mp 1)^{r+1} \left( \tilde{\zeta}\left(\mp(l+1)\right) - \sum_{n_1=1}^{r} \frac{(\mp 1)^{n_1}}{n_1^{l+1}} \right).$$

*and*

$$I^{+}_{0,0,r,1} = \int_0^1 \frac{x^r}{1 + x} dx = (-1)^{r+1} \left( \tilde{\zeta}(-1) - \sum_{n_1=1}^{r} \frac{(-1)^{n_1}}{n_1} \right).$$

**Theorem 10.** *Let $k$ $r$,and $l$ be integers such that, $k, r \geq 0$, $l \geq 1$. Then:*

$$I^{\pm}_{k,l,r,0} = \int_0^1 (-\log(1 \pm x))^k \cdot x^r (-\log x)^l dx$$
$$= k!l! \sum_{n_1>n_2>n_3>\ldots>n_k\geq 1} \frac{(\mp 1)^{n_1}}{(n_1+r+1)^{l+1} \cdot n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k},$$

*and*

$$\begin{aligned} I^{+}_{k,0,r,0} &= \int_0^1 \left(-\log\left(1+x\right)\right)^k \cdot x^r dx \\ &= k! \sum_{n_1>n_2>n_3>\ldots>n_k\geq 1} \frac{(-1)^{n_1}}{(n_1+r+1)\cdot n_1\cdot n_2\cdot n_3\cdot\ldots\cdot n_k}. \end{aligned}$$

*In particular*

$$I^{+}_{1,0,r,0} = -\int_0^1 \log\left(1+x\right)\cdot x^r dx = \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1}}{(n_1+r+1)\cdot n_1}$$
$$= \begin{cases} \frac{2}{r+1}\log\left(2\right)+S_r & \text{if} \quad r=2p \\ -S_r & \text{if} \quad r=2p+1 \end{cases},$$

*where* $S_r = \frac{1}{r+1}\sum_{i=1}^{r+1}\frac{(-1)^i}{i}$ *and* $p \in \mathbb{Z}$.

**Theorem 11.** *Let* $k, r, l$ *and* $m$ *be integers with,* $k, l, r \geq 0$; $m$, $l \geq 1$. *Then:*

$$I^{-}_{k,l,r,m} = \int_0^1 \frac{\left(-\log\left(1-x\right)\right)^k}{\left(1-x\right)^m}\cdot x^r \left(-\log x\right)^l dx = \sum_{i=0}^{r} \left(-1\right)^i \binom{r}{i} I^{-}_{l,k,i-m,0.}$$

**Corollary 12.** *If* $l$ *is an integer,* $l \geq 1$, *then:*

$$I^{-}_{1,l,0,0} = -\int_0^1 \log\left(1-x\right)\cdot\left(-\log x\right)^l dx = l! \sum_{n_1=1}^{\infty} \frac{1}{\left(n_1+1\right)^{l+1}\cdot n_1}$$
$$= l!\left(l+1-\sum_{i=2}^{l+1}\zeta\left(i\right)\right),$$

*while for* $l \geq 0$ :

$$I^{+}_{1,l,0,0} = -\int_0^1 \log\left(1+x\right)\cdot\left(-\log x\right)^l dx = -l! \sum_{n_1=1}^{\infty} \frac{(-1)^{n_1}}{\left(n_1+1\right)^{l+1}\cdot n_1}$$
$$= l!\left(l+1+\tilde{\zeta}\left(-1\right)+\sum_{i=1}^{l+1}\tilde{\zeta}\left(-i\right)\right).$$

**Theorem 13.** *Let* $l, m, r, k$ *be integers such that,* $l$, $m$, $r \geq 1$, $k \geq 2$. *Then:*

$$J^{\pm}_{l,r,m,k} = \int_0^1 \frac{x^{kr+k-1}}{\left(1\pm x^k\right)^m}\cdot\left(-\log x\right)^l dx = \frac{1}{k^{l+1}} I^{\pm}_{0,l,r,m.}$$

**Corollary 14.** *If* $l$ *is an integer,* $l \geq 1$. *Then:*

$$J^{\pm}_{l,0,1,2} = \int_0^1 \frac{x}{1\pm x^2}\cdot\left(-\log x\right)^l dx = \frac{1}{2^{l+1}} I^{\pm}_{0,l,0,1} = \mp\frac{1}{2^{l+1}} l!\tilde{\zeta}\left(\mp\left(l+1\right)\right).$$

**Theorem 15.** *Let $r$ ,$l$, $m_1$ and $m_2$ be integers with, $l, m_1, m_2 \geq 0$. Then:*

$$I_{l,r,m_1,m_2} = \int_0^1 \frac{x^r (-\log x)^l}{(1-x)^{m_1} (1+x)^{m_2}} dx = \sum_{i=1}^{m_1} A_i I^-_{0,l,r,i} + \sum_{j=1}^{m_2} B_j I^+_{0,l,r,j} \ ,$$

*where $A_i, B_j \in \mathbb{R}$, $i \in \{1, 2, ..., m_1\}$, $j \in \{1, 2, ..., m_2\}$.*

**Corollary 16.** *If $l$ is an integer, $l \geq 1$. Then:*

$$\begin{aligned} I_{l,0,1,1} = \int_0^1 \frac{1}{1-x^2} \cdot (-\log x)^l \, dx &= \frac{1}{2} \left( I^-_{0,l,0,1} + I^+_{0,l,0,1} \right) \\ &= \frac{1}{2} l! \left( \zeta(l+1) - \tilde{\zeta}(-l-1) \right) \\ &= l! \left( 1 - \frac{1}{2^{l+1}} \right) \zeta(l+1) \, . \end{aligned}$$

**Corollary 17.** *If $l$ is an integer, $l \geq 1$. Then:*

$$\begin{aligned} I_{l,1,1,1} = \int_0^1 \frac{x}{1-x^2} \cdot (-\log x)^l \, dx &= \frac{1}{2} \left( I^-_{0,l,1,1} + I^+_{0,l,1,1} \right) \\ &= \frac{1}{2} l! \left( \zeta(l+1) + \tilde{\zeta}(-l-1) \right) = \frac{1}{2^{l+1}} l! \zeta(l+1) \, . \end{aligned}$$

With the help on this type of integrals there can be calculated, using Maple, quickly, and with high precision, the values $\widetilde{\zeta}_{k+1}(\pm(l+1), \{1\}_k)$, $k,\ l \in \mathbb{Z}$, $k > 0, l > 1$. For example,

$$(14) \quad \zeta_4(6,1,1,1) = \frac{(-1)^{3+5}}{\Gamma(4)\Gamma(6)} \int_0^1 \frac{\log^3(1-x)}{1-x} \cdot \log^5 x dx \approx .0001060902289102175205140559549145517589881...$$

and

$$(15) \quad \tilde{\zeta}_4(-6,1,1,1) = \frac{(-1)^{3+5+1}}{\Gamma(4)\Gamma(6)} \int_0^1 \frac{\log^3(1+x)}{1+x} \cdot \log^5 x dx \approx .0000237218053494050919268700955137632799973 83...$$

An interesting application of integrals from this subchapter is the one studied by Sondow and Sergey Zlobin, with the help of which they discovered calculation formulae for some integrals on polytopes.

## 5. The calculation of some sums

In this last chapter I analyzed several types of series: *the BBP-Ramanujan type sums* in connection with *the hypergeometrical* series, *the Euler type sums* and *the Kempner-Irwin type harmonic subseries.*

### 5.1. The BBP-Ramanujan type sums

We reunited the two types of series, *tip BBP type* and *Ramanujan type*, for which we grouped several demonstration methods: using the Machin identities, BBP series which sums are logarithms or using the identities which contain values of the polylogarithmic function.

**Definition 18.** *We call* series of BBP-Ramanujan type , *the hypergeometric series of the following form*

$$\sum_{n=0}^{\infty} u^n x^n \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \cdot \frac{P(n)}{Q(n)}, \tag{16}$$

*where* $u = 1$ *or* $u = -1$; $x, a_i, b_j \in \mathbb{C}$, $i \in \{1, 2, ..., p\}$ , $j \in \{1, 2, ..., q\}$ ; $P, Q \in \mathbb{C}[X]$ , $Q$ *does not have positive integer roots and with* L. A. Pochhammer symbols*:*

$$(a)_n = a(a+1) \cdot \dots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{17}$$

*We call the number* $x$ base.

These series can be discovered using PSLQ algorithm. For the binomial extension of BBP series there are integrals with the same sum, which can be found using the computer with the same PSLQ algorithm. I presented the algorithm used for obtaining some constants in a given base, starting from a given position of the numbers after the coma.

### 5.2. Hypergeometrical series

The hypergeometric series have been studied by many mathematicians (Mary Celine Fasenmyler, R. W. Jr. Gosper, Doron Zeilbenger, etc.) building various algorithms of calculation which can be found implemented into the EKHAD package, made in Maple. These are particular cases of BBP-Ramanujan series.

Using the hypergeometric series there can be obtained expression of the BBP-Ramanujan type series. Thus. applying I Maple for the BBP sum or for the Apéry binomial series

```
>sum((4/(8*n+1)-2/(8*n+4)-1/(8*n+5)-1/(8*n+6))/(16^n), n=0..infinity);
>(5/2)*sum(((-1)^(n-1))/(binomial(2*n,n)*n^3), n=1..infinity);
```

we obtain

$$\begin{aligned} \pi &= \sum\nolimits_{k=1}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right) \\ &= \frac{47}{15}\,{}_5F_4\left[\begin{matrix} 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{1}{8} \\ \frac{7}{4}, \frac{13}{8}, \frac{3}{2}, \frac{9}{8} \end{matrix} ; \frac{1}{16}\right] + \frac{271}{39312}\,{}_5F_4\left[\begin{matrix} 2, \frac{7}{4}, \frac{13}{8}, \frac{3}{2}, \frac{9}{8} \\ \frac{17}{8}, \frac{11}{4}, \frac{21}{8}, \frac{5}{2} \end{matrix} ; \frac{1}{16}\right] \\ &\quad + \frac{1}{20944}\,{}_5F_4\left[\begin{matrix} 3, \frac{17}{8}, \frac{11}{4}, \frac{21}{8}, \frac{5}{2} \\ \frac{15}{4}, \frac{29}{8}, \frac{7}{2}, \frac{25}{8} \end{matrix} ; \frac{1}{16}\right]; \end{aligned} \tag{18}$$

(19) $$\zeta(3)=\frac{5}{2}\sum\nolimits_{n=1}^{\infty}\frac{(-1)^{n-1}}{\binom{2n}{n}n^3}=\frac{5}{4}\,{}_4F_3\left[\begin{matrix}1,1,1,1\\2,2,3/2\end{matrix};-\frac{1}{4}\right].$$

### 5.3. Euler type sums

The last two subchapters refer to two types of series with slow convergence. The first type, *the Euler type sums*, is made up of double series which contain harmonic numbers. Due to the slow convergence for the calculation as precise as possible there is needed a long time and computers with high power of calculation. Therefore, *Jonathan M. Borwein*, *David H. Bailey*, *Roland Girgensohn*, succeeded to find various expressions for these formulae using the PSLQ algorithm. In the present paper it is given a method of demonstration based on a function called *kernel* and on *Cauchy theorem of residues*.

### 5.4. Kempner-Irwin type series

The series that I called *tip Kempner-Irwin type*, are subseries of the harmonic one. The terms of these subseries are the inverses of some numbers which respect a condition upon the numbers from which are made up in a given base. Part of the result in this chapter can be found in [4].

**Definition 19.** *Let $X_m$ be a string of $m \geq 0$ digits. We denote by $S^-$ the set of all the positive integers that* do not contain *$X_m$, in their decimal representation, by $S^+$ we denote the set of all positive integers that* contain *$X_m$, and by $S^{(p)}$ the set of all positive integers that* contain *$X_m$* exactly *$p$ times. We will also make use of the set $S^{(\leq p)}$, of all positive integers that* contain *$X_m$* no more *than $p$ times and $S^{(\geq p)}$, of all positive integers that* contain *$X_m$* at least *$p$ times.*

*The* Kempner type *series are the series of one of the following form:*

(20) $$\Psi^-_{k;X_m} = \sum_{s\in S^-}\frac{1}{s^k},\quad \Psi^+_{k;X_m} = \sum_{s\in S^+}\frac{1}{s^k},$$

(21) $$\Psi^{(p)}_{k;X_m} = \sum_{s\in S^{(p)}}\frac{1}{s^k},\quad \Psi^{(\leq p)}_{k;X_m} = \sum_{s\in S^{(\leq p)}}\frac{1}{s^k},$$

(22) $$\Psi^{(\geq p)}_{k;X_m} = \sum_{s\in S^{(\geq p)}}\frac{1}{s^k},\qquad \text{where } k\in\mathbb{N}^*,$$

*If we consider a set $X$, of strings of digits, and a set $S$, of numbers that meet different conditions, expressed in on of the five forms above, The series*

(23) $$\Psi_{k;X} = \sum_{s\in S}\frac{1}{s^k},\qquad \text{where } k\in\mathbb{N}^*,$$

*we call* Irwin type.

It is known that the harmonic series is divergent. Despite all this some Kempner-Irwin series are convergent (for example $\Psi^-_{1;"9"} = \sum_{s\in S^-} 1/s$, $\Psi^{(p)}_{1;"9"} = \sum_{s\in S^{(p)}} 1/s$,

$\Psi_{1;"9"}^{(\leq p)} = \sum_{s\in S^{(\leq p)}} 1/s)$. For the calculation of these series Robert Baillie made an algorithm implemented in *Mathematica*. It is remarkable the fact that in the binary base these types of series have a rapid convergence and can be easily calculated and with a high precision in *Maple*. I extended the Kempner-Irwin series to Kempner-Irwin series in the logarithmic form, as they are also convergent, as for example

$$\Omega_{1;"9"}^{-} = \sum_{s\in S^{-}} \frac{1}{s\ln s} \simeq 3.41067..., \tag{24}$$

where $S^{-}$ is the set of all positive integer numbers which do not contain the digit 9.

Toward the end of the chapter it is studied the convergence of some strings of Kempner-Irwin type sums:

**Theorem 20.** a) *Let $X_m$ a string with m digits having period p, i.e.*

$$X_m = "\underbrace{d_1d_2...d_pd_1d_2...d_p...d_1d_2...d_p}_{m=kp \text{ digits}}". \tag{25}$$

*Let $\Psi_{X_m}^{-}$ sum of $1/s$ where $s$ does not contain a string $X_m$. Then*

$$\lim_{m\to\infty} \frac{\Psi_{1;X_m}^{-}}{10^m} = \frac{10^p}{10^p-1}\log 10. \tag{26}$$

b) *Let $X_m = "d_1d_2...d_m"$ a string with m digits non-periodical. Let $\Psi_{X_m}^{-}$ sum of $1/s$ where $s$ does not contain a string $X_m$. Then*

$$\lim_{m\to\infty} \frac{\Psi_{1;X_m}^{-}}{10^m} = \log 10. \tag{27}$$

**Theorem 21.** a) *Let $X_m$ a string with m digits having period p, i.e.*

$$X_m = "\underbrace{d_1d_2...d_pd_1d_2...d_p...d_1d_2...d_p}_{m=kp \text{ digits}}". \tag{28}$$

*Let $\Omega_{X_m}^{-}$ sum of $1/(s\ln s)$ where $s$ does not contain a string $X_m$. Then*

$$\lim_{m\to\infty} \frac{\Omega_{1;X_m}^{-}}{10^m \ln\frac{m}{m-1}} = \frac{10^p}{10^p-1}. \tag{29}$$

b) *Let $X_m = "d_1d_2...d_m"$ a string with m digits non-periodical. Let $\Omega_{X_m}^{-}$ sum of $1/(s\ln s)$ where $s$ does not contain a string $X_m$. Then*

$$\lim_{m\to\infty} \frac{\Omega_{1;X_m}^{-}}{10^m \ln\frac{m}{m-1}} = \frac{10^p}{10^p-1} = 1. \tag{30}$$

The folowing theorem is demonstrated in [4]:

**Theorem 22.** *The string $\left(\Psi_{1;"89"}^{(r)}\right)_{r\in\mathbb{N}^*}$ decreases and $\lim_{r\to\infty}\Psi_{1;"89"}^{(r)} \simeq 22.2176459...$, where*

$$\Psi_{1;"89"}^{(r)} = \sum_{s\in S^{(r)}} \frac{1}{s} \tag{31}$$

*and $S^{(r)}$ is the set of positive integer numbers which end in 8 'and contain "89" of exactly $r$ times.*

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