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LOCALIZATION IN THE CATEGORY OF MTL - ALGEBRAS

SUMMARY OF THE PH. D. THESIS

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Introduction

In the last years, residuated structures became popular in computer science since it was understood that they play a fundamental role in fuzzy logics. We recall that the origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull ([**70**]), Dilworth ([**38**]), Ward and Dilworth ([**97**]), Ward ([**96**]), Balbes and Dwinger ([**2**]) and Pavelka ([**80**]).

Apart from their logical interest, residuated lattices have interesting algebraic properties (see [**7**], [**21**], [**38**], [**47**], [**69**], [**79**], [**96**], [**97**]). As Idziak proved in [**58**], that the class of residuated lattices is equational. In literature, these lattices have been known under many names: *BCK- latices* in [**52**], *full BCK- algebras* in [**70**], FL_{ew}*- algebras* in [**78**], and *integral, residuated, commutative l-monoids* in [**8**].

Basic Fuzzy logic (BL from now on) is the many-valued residuated logic introduced by Hájek in [**52**] to cope with the logic of continuous t-norms and their residua. Monoidal logic (ML from now on), introduced by Hőhle ([**56**]), is a logic whose algebraic counterpart is the class of residuated lattices; MTL algebras (see [**41**]) are algebraic structures for the Esteva-Godo monoidal t-norm based logic (MTL), a many-valued propositional calculus that formalizes the structure of the real unit interval $[0, 1]$, induced by a left–continuous t-norm. MTL algebras were independently introduced in [**43**] under the name *weak-BL algebras.*

We recall ([**43**]) that a non-commutative residuated lattice (called sometimes *pseudo-residuated lattice, biresiduated lattice or generalized residuated lattice*) is an algebra $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 2, 0, 0)$ satisfying the following conditions: $(A, \wedge, \vee, 0, 1)$ is a bounded lattice; $(A, \odot, 1)$ is a monoid and $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any $x, y, z \in A$.

Pseudo $BL-$ *algebras* were introduced in [**39**] as a non-commutative extension of Hájek's $BL-$algebras. Pseudo $BL-$algebras are bounded non-commutative residuated lattices $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ which satisfy the *pseudo-divisibility condition* $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$ and the *pseudo-prelinearity condition* $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$. Depending on the above conditions, there are two directions to extend pseudo $BL-$algebras. One direction investigates the (bounded) non-commutative residuated lattices satisfying the pseudo-divisibility condition which were studied under the name *(bounded) divisible pseudo - residuated lattices* or *bounded Rl - monoids.* The second direction deals with (bounded) non-commutative residuated lattices with the pseudo-prelinearity condition, that is, *pseudo* $MTL-$ *algebras.*

So, pseudo $MTL-$ algebras are non-commutative fuzzy structures which arise from pseudo t-norms, namely, pseudo $BL-$algebras without the pseudo-divisibility condition.

The main aim of this thesis is to characterize the lattice of congruence filters for a residuated lattice, the archimedean and hyperarchimedean residuated lattices and *to develop a theory of localization for MTL -algebras and to extend this theory to the non-commutative case of pseudo MTL-algebras (using the model of localization for BL and pseudo BL-algebras, see* [**18**] *and* [**20**]*).*

A remarkable construction in ring theory is the *localization ring* $A_{\mathcal{F}}$ associated with a Gabriel topology $\mathcal{F}$ on a ring A (see [**45**]); for certain issues connected to the therm *localization* we have in view Chapter IV : *Localization* in N. Popescu' s book [**86**].

In Lambek' s book [**74**] it is introduced the notion of *complete ring of quotients* of a commutative ring, as a particular case of localization ring (relative to the *dense ideals*).

Starting from the example of the ring, J. Schmid introduces in [**90**], [**91**] the notion of *maximal lattice of quotients* for a distributive lattice. The central role in this construction is played by the concept of *multipliers,* defined by W. H. Cornish in [**32**], [**33**].

Using the model of localization ring (see [**45**]), in [**48**] is defined for a bounded distributive lattice L the *localization lattice* $L_{\mathcal{F}}$ *of* L *with respect to a topology* $\mathcal{F}$ *on* L and is proved that the maximal lattice of quotients for a distributive lattice is a lattice of localization (relative to the topology of *regular* ideals).

The same theory is also valid for the *lattice of fractions* of a distributive lattice with 0 and 1 relative to an $\wedge$-closed system (see [**10**]).

A theory of localization for Hilbert and Hertz algebras was developed in [**12**]; for the case of Heyting algebras see [**34**], for the case of MV and pseudo MV-algebras see [**14**], [**22**], [**82**], for the case of BL and pseudo BL-algebras see [**18**], [**20**], [**82**] and for LMn-algebras see [**25**].

For some informal explanations of the notion of *localization* see [**76**], [**87**], [**88**].

All original results from this thesis are contained in [**23**], [**63**], [**64**] şi [**83**]-[**85**].

Chapter 1 : Reziduated lattices

The first chapter contains some results relative to residuated lattices.

All original results are contained in [**23**], [**64**] şi [**85**].

A *residuated lattice* is an algebra

$$(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$$

of type (2,2,2,2,0,0) equipped with an order $\leq$ satisfying the following:

(LR_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
(LR_2) $(A, \odot, 1)$ is a commutative ordered monoid;
(LR_3) $\odot$ and $\rightarrow$ form an adjoint pair, i.e. $c \leq a \rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in A$.

In this thesis, the symbols $\Rightarrow$ and $\Leftrightarrow$ are used for logical implication and logical equivalence.

Łukasiewicz structure, Produs structure, Gődel structure and Boolean algebras are examples of residuated lattice. Also, we present in my thesis two examples of residuated lattices which are not distributive lattices.

The class $\mathcal{RL}$ of residuated lattices is equational (see, [**58**]).

A residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called *BL-algebra*, if the following two identities hold in A :

(BL_1) $x \odot (x \rightarrow y) = x \wedge y$ (divisibility);
(BL_2) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

Łukasiewicz structure, Gődel structure and Product structure are $BL-$ algebras but not every residuated lattice, however, is a BL-algebra (see [**93**], p.16).

A residuated lattice $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is an MV-algebra iff it satisfies the additional condition: $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for any $x, y \in A$.

We denote by $B(A)$ the set of all complemented elements of the lattice $L(A) = (A, \wedge, \vee, 0, 1)$, where A is a residuated lattice.

We recall the basic definitions, examples and rules of calculus in a residuated lattice and we prove new results about these algebras. Also we present the connection between residuated lattices and Hilbert algebras.

For a residuated lattice A We denote by $Ds(A)$ the set of all congruence filters (deductive systems) of A. The lattice $(Ds(A), \subseteq)$ is a complete Brouwerian lattice (hence distributive), the compacts elements being exactly the principal **ds** of A.

For two deductive systems D_1, $D_2 \in Ds(A)$ we define

$$D_1 \rightarrow D_2 = \{a \in A : D_1 \cap [a) \subseteq D_2\},$$

so, $(Ds(A), \vee, \wedge, \rightarrow, \{1\}, A)$ become a Heyting algebra, , where for $D \in Ds(A)$,

$$D^* = D \rightarrow \mathbf{0} = D \rightarrow \{1\} = \{x \in A : x \vee y = 1, \text{ for every } y \in D\}.$$

Theorem 1.39 characterizes the residuated lattices for which the lattice of congruence filters is a Boolean algebra :

If A is a residuated lattice, then the following assertions are equivalent:

(i) $(Ds(A), \vee, \wedge, ^*, \{1\}, A)$ *is a Boolean algebra;*

(ii) *Every* **ds** *of A is principal and for every $a \in A$ there exists $n \geq 1$ such that* $a \vee (a^n)^* = 1$.

For the distributive lattice $Ds(A)$ we denote by $Spec(A)$ the set of all prime deductive systems of A and by $Irc(A)$ the set of all completely meet-irreducible elements of $Ds(A)$.

In my thesis we put in evidence new characterizations for the elements of $Spec(A)$ and $Irc(A)$ in Proposition 1.46., Corollary 1.47., 1.48. and Theorem 1.49. :

For a proper deductive system $P \in Ds(A)$ the following conditions are equivalent:

(i) $P \in Spec(A)$,

(ii) *Pentru orice $x, y \in A \backslash P$ there is $z \in A \backslash P$such that $x \leq z$ and $y \leq z$,*

(iii) *If $x, y \in A$ and $< x > \cap < y > \subseteq P$, then $x \in P$ or $y \in P$*

(iv) *For every $x, y \in A/P, x \neq 1, y \neq 1$, there is $z \in A/P, z \neq 1$ such that $x \leq z$ and$y \leq z$,*

(v) *For every $D \in Ds(A), D \rightarrow P = P$ or $D \subseteq P$.*

Relative to the uniqueness of deductive systems as intersection of primes we prove that this is possible only in the case of Boolean algebras :

Theorem 1.51. *If every $D \in Ds(A)$ has an unique representation as an intersection of elements of $Spec(A)$, then $(Ds(A), \vee, \wedge, ^*, \{1\}, A)$ is a Boolean algebra.*

A deductive system $D \in Ds(A)$, $D \neq A$ is called *maximal relative to a* if $a \notin D$ and if $D' \in Ds(A)$ is proper such that $a \notin D'$, and $D \subseteq D'$, then $D = D'$.

About these deductive systems we have the following results:

Corollary 1.53. *For any $a \in A, a \neq 1$, there is a deductive system D_a maximal relative to a.*

Theorem 1.54. *For $D \in Ds(A), D \neq A$ the following are equivalent:*

(i) $D \in Irc(A)$;

(ii) *There is $a \in A$ such that D is maximal relative to a.*

Theorem 1.55. *Let $D \in Ds(A)$ be a deductive system , $D \neq A$ and $a \in A \backslash D$. Then the following conditions are equivalent:*

(i) *D is maximal relative to a*;

(ii) *For every $x \in A \backslash D$ there is $n \geq 1$ such that $x^n \rightarrow a \in D$.*

and

Corollary 1.56. *For $D \in Ds(A), D \neq A$ the following conditions are equivalent:*

(i) $D \in Irc(A)$;

(ii) *In the set $A/D \backslash \{\mathbf{1}\}$ we have an element $p \neq \mathbf{1}$ with the property that for every $\alpha \in A/D \backslash \{\mathbf{1}\}$ there is $n \geq 1$ such that $\alpha^n \leq p$.*

A deductive system P of A is a *minimal prime* if $P \in Spec(A)$ and, whenever $Q \in Spec(A)$ and $Q \subseteq P$, we have $P = Q$.

We obtain that

Propozition 1.57. *If P is a minimal prime deductive system, then for any $a \in P$ there is $b \in A \backslash P$ such that $a \vee b = 1$.*

An element a of a residuated lattice A is called *infinitesimal* if $a \neq 1$ and $a^n \geq a^*$ for any $n \geq 1$. We denote by $Inf(A)$ the set of all infinitesimals of A and by $Rad(A)$ the intersection of the maximal deductive systems of A.

We obtain

Corollary 1.66. $Inf(A) \subseteq Rad(A)$.

We introduce the notions of *archimedean* and *hyperarchimedean* residuated lattice and we prove a theorem of *Nachbin* type for residuated lattices.

A residuated lattice A is called *archimedean* if one of the equivalent conditions from Lemma l.67 is satisfied:

Lemma 1.67. *In any residuated lattice A the following are equivalent:*

(i) *For every $a \in A, a^n \geq a^*$ for any $n \geq 1$ implies $a = 1$;*
(ii) *For every $a, b \in A, a^n \geq b^*$ for any $n \geq 1$ implies $a \vee b = 1$;*
(iii) *For every $a, b \in A, a^n \geq b^*$ for any $n \geq 1$ implies $a \to b = b$ and $b \to a = a$.*

One can easily remark that a residuated lattice is archimedean iff it has no infinitesimals.

An element $a \in A$ is called *archimedean* if it satisfy the condition: there is $n \geq 1$ such that $a^n \in B(A)$(equivalent with $a \vee (a^n)^* = 1$). A residuated lattice A is called *hyperarchimedean* if all its elements are archimedean. Every hyperarchimedean residuated lattice is archimedean.

For a residuated lattice A, if A is hyperarchimedean, then for any deductive system D, the quotient residuated lattice A/D is archimedean.

We have the following result:

Theorem 1.70. *For a residuated lattice A the following conditions are equivalent:*

(i) *A is hyperarchimedean;*
(ii) *$Spec(A) = Max(A)$;*
(iii) *Any prime deductive system is minimal prime.*

and a theorem of *Nachbin* type for residuated lattices :

Theorema 1.71. *For a residuated lattice A the following assertions are equivalent:*

(i) *A is hyperarchimedean;*
(ii) *$(Spec(A), \subseteq)$ is unordered.*

Chapter 2 : Localization of MTL -algebras

In this chapter we develop a theory of localization for MTL-algebras.

The original results are contained in [**83**] and [**84**].

We recall that a MTL *algebra* is a residuated lattice satisfying the preliniarity equation:

(MTL) $(x \to y) \vee (y \to x) = 1.$

Every linearly ordered residuated lattice is a $MTL-$ algebra. A $MTL-$ algebra A is a $BL-$ algebra iff in A is verified the divisibility condition: $x \odot (x \to y) = x \wedge y$. So, $BL-$ algebras are examples of $MTL-$ algebras,

We recall the basic definitions, examples and rules of calculus in $MTL-$ algebras and we prove new results about these algebras.

For a $MTL-$ algebra A we denote by $B(A)$ the boolean center of A.

In this chapter for a MTL-algebra A we introduce the notion of MTL-algebra of fractions relative to a $\wedge$-closed system .

For a $\wedge$-*close system* $S \subseteq A$ ($1 \in S$ and $x, y \in S$ implies $x \wedge y \in S$) we consider the congruence relation θ_S defined by :

$$(x, y) \in \theta_S \text{ if and only if there is } e \in S \cap B(A) \text{ such that } x \wedge e = y \wedge e.$$

For $x \in A$ we denote by x/S the equivalence class of x relative to θ_S and by $A[S] = A/\theta_S$. In $A[S]$, $\mathbf{0} = 0/S$, $\mathbf{1} = 1/S$ and for every $x, y \in A$,

$$x/S \wedge y/S = (x \wedge y)/S, x/S \vee y/S = (x \vee y)/S$$

$$x/S \odot y/S = (x \odot y)/S, x/S \to y/S = (x \to y)/S.$$

Then we have :

Theorem 2.5. *If A' is a MTL-algebra and $f : A \to A'$ a morphism of MTL-algebras such that $f(S \cap B(A)) = \{\mathbf{1}\}$, then there is a unique morphism of MTL-algebras $f' : A[S] \to A'$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "p_S"] \arrow[dr, "f"'] & A[S] \arrow[d, "f'"] \\ & A' \end{tikzcd}$$

is commutative (i.e. $f' \circ p_S = f$), unde $p_S : A \to A[S]$ is an ontomorphism of $MTL-$ algebras.

This theorem allows us to call $A[S]$ the *MTL-algebra of fractions relative to the $\wedge-$closed system S.*

If the MTL-algebra A is a $BL-$ algebra then $A[S]$ is also a $BL-$ algebra.

Starting from the model of J. Schmid ([**90**], [**91**]) we introduce the notion of *maximal MTL-algebra of quotients* for a $MTL-$ algebra using the *strong multipliers.*

We denote by $I(A)$ the set of all *ordered ideals* of $MTL-$ algebra $L(A)$:

$$I(A)=\{I\subseteq A: \text{if } x,y\in A, x\leq y \text{and } y\in I, \text{then } x\in I\}.$$

By *partial strong multiplier* on A we mean a map $f:I\rightarrow A$, where $I\in I(A)$, which verifies the next axioms:

$(smMTL_1)$ $f(e\odot x)=e\odot f(x)$, for every $e\in B(A)$ and $x\in I$;
$(smMTL_2)$ $x\odot(x\rightarrow f(x))=f(x)$, for every $x\in I$;
$(smMTL_3)$ If $e\in I\cap B(A)$, then $f(e)\in B(A)$;
$(smMTL_4)$ $x\wedge f(e)=e\wedge f(x)$, for every $e\in I\cap B(A)$ and $x\in I$.
(we remark that $e\odot x\in I$ since $e\odot x\leq e\wedge x\leq x$).

By $smMTL_2$ we deduce $(smMTL_5)$: $f(x)\leq x$, for every $x\in I$;

If A is a BL algebra, then $smMTL_2$ is a consequence of $smMTL_5$ (because in this case $x\odot(x\rightarrow f(x))\overset{smMTL_5}{=}f(x)$, for every $x\in I$).

By $dom(f)\in I(A)$ we denote the domain of f; if $dom(f)=A$, we call f *total.* To simplify the language, we will use *strong multiplier* instead *partial strong multiplier,* using *total* to indicate that the domain of a certain strong multiplier is A.

The maps $\mathbf{0},\mathbf{1}:A\rightarrow A$ defined by $\mathbf{0}(x)=0$ and $\mathbf{1}(x)=x$, for every $x\in A$ are strong multipliers on A. For $a\in B(A)$ and $I\in I(A)$ the map $f_a:I\rightarrow A$ defined by $f_a(x)=a\wedge x$, for every $x\in I$ is a strong multiplier on A (called *principal*). If $dom(f_a)=A$, we denote f_a by $\overline{f_a}$.

For $I\in I(A)$, we denote

$$M(I,A)=\{f:I\rightarrow A \mid f \text{ is a strong multiplier on } A\}$$

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$$M(A)=\bigcup_{I\in I(A)}M(I,A).$$

If $I_1,I_2\in I(A)$ and $f_i\in M(I_i,A), i=1,2$, then

$f_1(x)\odot[x\rightarrow f_2(x)]=f_2(x)\odot[x\rightarrow f_1(x)]$, for every $x\in I_1\cap I_2$.

For $I_1,I_2\in I(A)$ and $f_i\in M(I_i,A), i=1,2$, we define $f_1\wedge f_2$, $f_1\vee f_2$, $f_1\otimes f_2$, $f_1\rightsquigarrow f_2:I_1\cap I_2\rightarrow A$ by

$$(f_1\wedge f_2)(x)=f_1(x)\wedge f_2(x),$$

$$(f_1\vee f_2)(x)=f_1(x)\vee f_2(x),$$

$$(f_1\otimes f_2)(x)=f_1(x)\odot[x\rightarrow f_2(x)]\overset{mtl-c_7}{=}f_2(x)\odot[x\rightarrow f_1(x)],$$

$$(f_1\rightsquigarrow f_2)(x)=x\odot[f_1(x)\rightarrow f_2(x)],$$

for every $x\in I_1\cap I_2$.

We obtain :

Proposition 2.14. $(M(A),\wedge,\vee,\otimes,\rightsquigarrow,\mathbf{0},\mathbf{1})$ *is a MTL-algebra.*

The map $v_A:B(A)\rightarrow M(A)$ defined by $v_A(a)=\overline{f_a}$ for every $a\in B(A)$ is a monomorphism of MTL-algebras.

A nonempty set $I\subseteq A$ is called *regular* if for every $x,y\in A$ such that $x\wedge e=y\wedge e$ for every $e\in I\cap B(A)$, then $x=y$.

We denote by

$$M_r(A)=\{f\in M(A): dom(f)\in I(A)\cap R(A)\},$$

where

$$R(A)=\{I\subseteq A: I \text{ is a regular subset of} A\}.$$

On $M_r(A)$ we consider the congruence relation ρ_A defined by

$(f_1, f_2) \in \rho_A$ if and only if f_1şi f_2 coincide on the intersection of their domains.

For $f \in M_r(A)$ with $I = dom(f) \in I(A) \cap R(A)$, we denote by $[f, I]$ the equivalence class of f modulo ρ_A and by

$A'' = M_r(A)/\rho_A$, which is a Boolean algebra (see Remark 2.14.).

Let the map $\overline{v_A} : B(A) \to A''$ defined by $\overline{v_A}(a) = [\overline{f_a}, A]$ for every $a \in B(A)$. Then:

(i) $\overline{v_A}$ is a monomorphism of MTL algebras;
(ii) For every $a \in B(A)$, $[\overline{f_a}, A] \in B(A'')$;
(iii) $\overline{v_A}(B(A)) \in R(A'')$.

Since for every $a \in B(A)$, $\overline{f_a}$ is the unique maximal strong multiplier on $[\overline{f_a}, A]$ we can identify $[\overline{f_a}, A]$ with $\overline{f_a}$. So, since $\overline{v_A}$ is injective map, the elements of $B(A)$ can be identified with the elements of the set $\{\ \overline{f_a} : a \in B(A)\}$.

We introduce the notions of *MTL-algebra of fractions* and *maximal MTL-algebra of quotients* .

If A is a MTL algebra, a MTL algebra F is called *MTLalgebra of fractions of* A if:

$(fr - MTL_1)$ $B(A)$ is a MTL subalgebra of F;
$(fr - MTL_2)$ For every $a', b', c' \in F, a' \neq b'$, there exists $e \in B(A)$ such that $e \wedge a' \neq e \wedge b'$ and $e \wedge c' \in B(A)$.

As a notational convenience, we write $A \preceq F$ to indicate that F is a MTL algebra of fractions of A.

$Q(A)$ is the *maximal MTL algebra of quotients* of A if $A \preceq Q(A)$ and for every MTL algebra F with $A \preceq F$ there exists a monomorphism of MTL algebras $i : F \to Q(A)$.

If $A \preceq F$, then F is a Boolean algebra, so $Q(A)$ is also a Boolean algebra.

An important result is:

Theorem 2.25 A'' is the maximal MTL algebra $Q(A)$ of quotients of A.

We introduce on a MTL - algebra the notion of *topology* as in the case of rings or distributive lattices. We study the notions of *MTL - algebra of localization and strong MTL - algebra of localization $foraMTL-$ algebra A relative to the topology $\mathcal{F}$ on A*; We denote these by $A_{\mathcal{F}}$ and $s - A_{\mathcal{F}}$ and we prove that the maximal MTL - algebra of quotients and MTL - algebra of fraction relative to an $\wedge-$ closed system are strong MTL - algebras of localization.

We define the notion of *$\mathcal{F}-$ multiplier,* where $\mathcal{F}$ is a topology on a $MTL-$ algebra A.

A non-empty set $\mathcal{F}$ of elements $I \in I(A)$ will be called a *topology* on A if the following axioms hold:

(top_1) If $I_1 \in \mathcal{F}, I_2 \in I(A)$ and $I_1 \subseteq I_2$, then $I_2 \in \mathcal{F}$ (hence $A \in \mathcal{F}$);
(top_2) If $I_1, I_2 \in \mathcal{F}$, then $I_1 \cap I_2 \in \mathcal{F}$.

We will use the $\mathcal{F}$ -multipliers in the construction of *$MTL-$ algebra of localization* $A_{\mathcal{F}}$ relative to the topology $\mathcal{F}$.

We define the congruence relation $\theta_{\mathcal{F}}$ on A by:

$$(x, y) \in \theta_{\mathcal{F}} \Leftrightarrow \text{there is } I \in \mathcal{F} \text{ such that } e \wedge x = e \wedge y \text{ for every } e \in I \cap B(A).$$

A $\mathcal{F}-$ *multiplier* is a mapping $f : I \to A/\theta_{\mathcal{F}}$ where $I \in \mathcal{F}$ and for every $x \in I$ and $e \in B(A)$ the following axioms are fulfilled:

$(mMTL_1)$ $f(e \odot x) = e/\theta_{\mathcal{F}} \wedge f(x) = e/\theta_{\mathcal{F}} \odot f(x)$;
$(mMTL_2)$ $x/\theta_{\mathcal{F}} \odot (x/\theta_{\mathcal{F}} \to f(x)) = f(x)$.

We shall denote by $M(I, A/\theta_{\mathcal{F}})$ the set of all the $\mathcal{F}-$ multipliers having the domain $I \in \mathcal{F}$ and $M(A/\theta_{\mathcal{F}}) = \underset{I\in\mathcal{F}}{\cup} M(I, A/\theta_{\mathcal{F}})$. If $I_1, I_2 \in \mathcal{F}$, $I_1 \subseteq I_2$ we have a canonical mapping $\varphi_{I_1,I_2} : M(I_2, A/\theta_{\mathcal{F}}) \to M(I_1, A/\theta_{\mathcal{F}})$ defined by $\varphi_{I_1,I_2}(f) = f_{|I_1}$ for $f \in M(I_2, A/\theta_{\mathcal{F}})$. Let us consider the directed system of sets

$$\left\langle \{M(I, A/\theta_{\mathcal{F}})\}_{I\in\mathcal{F}}, \{\varphi_{I_1,I_2}\}_{I_1,I_2\in\mathcal{F},I_1\subseteq I_2} \right\rangle$$

and denote by $A_{\mathcal{F}}$ the inductive limit in the category of sets:

$$A_{\mathcal{F}} = \varinjlim_{I\in\mathcal{F}} M(I, A/\theta_{\mathcal{F}}).$$

For any $\mathcal{F}-$ multiplier $f : I \to A/\theta_{\mathcal{F}}$ we shall denote by $\widehat{(I, f)}$ the equivalence class of f in $A_{\mathcal{F}}$.

If $f_i : I_i \to A/\theta_{\mathcal{F}}$, $i = 1, 2$, are $\mathcal{F}-$ multipliers, then $\widehat{(I_1, f_1)} = \widehat{(I_2, f_2)}$ (în $A_{\mathcal{F}}$) if and only if there is $I \in \mathcal{F}$, $I \subseteq I_1 \cap I_2$ such that $f_{1|I} = f_{2|I}$.

Let $f_i : I_i \to A/\theta_{\mathcal{F}}$, (with $I_i \in \mathcal{F}$, $i = 1, 2$), $\mathcal{F}-$ multipliers. We consider the mappings $f_1 \wedge f_2, f_1 \vee f_2, f_1 \odot f_2, f_1 \to f_2 : I_1 \cap I_2 \to A/\theta_{\mathcal{F}}$ defined by

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x), (f_1 \vee f_2)(x) = f_1(x) \vee f_2(x),$$

$$(f_1 \odot f_2)(x) = f_1(x) \odot [x/\theta_{\mathcal{F}} \to f_2(x)] \overset{mtl-c_8}{=} f_2(x) \odot [x/\theta_{\mathcal{F}} \to f_1(x)],$$

$$(f_1 \to f_2)(x) = x/\theta_{\mathcal{F}} \odot [f_1(x) \to f_2(x)],$$

for every $x \in I_1 \cap I_2$, and let

$$\widehat{(I_1, f_1)} \curlywedge \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \wedge f_2), \widehat{(I_1, f_1)} \curlyvee \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \vee f_2),$$

$$\widehat{(I_1, f_1)} \otimes \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \odot f_2), \widehat{(I_1, f_1)} \longmapsto \widehat{(I_2, f_2)} = (I_1 \cap \widehat{I_2, f_1} \to f_2).$$

So, $(A_{\mathcal{F}}, \curlywedge, \curlyvee, \otimes, \longmapsto, \mathbf{0} = \widehat{(A, \mathbf{0})}, \mathbf{1} = \widehat{(A, \mathbf{1})})$ become a MTL-algebra called $MTL-$ *algebra of localization of* A *relative to the topology* $\mathcal{F}$.

To obtain the maximal MTL -algebra of quotients $Q(A)$ as a localization relative to a topology $\mathcal{F}$ we have to develop another theory of multipliers (meaning we add new axioms for $\mathcal{F}$-multipliers).

A *strong -* $\mathcal{F}-$ *multiplier* is a mapping $f : I \to A/\theta_{\mathcal{F}}$ (where $I \in \mathcal{F}$) which verifies the axioms $mMTL_1$, $mMTL_2$ and

$(mMTL_3)$ If $e \in I \cap B(A)$, then $f(e) \in B(A/\theta_{\mathcal{F}})$;
$(mMTL_4)$ $(x/\theta_{\mathcal{F}}) \wedge f(e) = (e/\theta_{\mathcal{F}}) \wedge f(x)$, for every $e \in I \cap B(A)$ and $x \in I$.

Analogous as in the case of $\mathcal{F}-$ multipliers if we work with strong-$\mathcal{F}-$ multipliers we obtain a $MTL-$ subalgebra of $A_{\mathcal{F}}$ denoted by $s - A_{\mathcal{F}}$ which will be called the strong-localization $MTL-$ algebra of A with respect to the topology $\mathcal{F}$.

We describe the localization MTL-algebra $A_{\mathcal{F}}$ in some special instances:

If we consider the Łukasiewicz structure $A = I = [0, 1]$ and the topology $\mathcal{F}(I) = \{I' \in \mathcal{I}(A) : I \subseteq I'\}$ then $A_{\mathcal{F}}$ is not a Boolean algebra.

In the case $\mathcal{F} = I(A) \cap R(A)$, $s - A_{\mathcal{F}}$ is exactly the maximal MTL-algebra $Q(A)$ of quotients of A which is a Boolean algebra

If $\mathcal{F}_S$ is the topology associated with a $\wedge-$closed system $S \subseteq A$, then the MTL-algebra $s - A_{\mathcal{F}_S}$ is isomorphic with $B(A[S])$.

Chapter 3 : Localization of Pseudo MTL - algebras

In this chapter we develop - taking as a guide-line the case of MTL -algebras - the theory of localization for pseudo MTL - algebras (which are non-commutative generalization of these). The main topic of this chapter is to generalize to pseudo $MTL-$ algebras the notions of $MTL-$ algebras of multipliers, $MTL-$ algebra of fractions and maximal $MTL-$ algebra of quotients. The structure, methods and techniques in this chapter are analogous to the structure, methods and techniques for $MTL-$ algebras exposed in Chapter 2.

We recall that a *pseudo MTL- algebra* ([**43**]) is an algebra $(A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2,2,2,2,2,0,0)$ equipped with an order $\leq$ satisfying the following axioms:

($psMTL_1$) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice relative to the order $\leq$;

($psMTL_2$) $(A, \odot, 1)$ is a monoid;

($psMTL_3$) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$, for every $x, y, z \in A$;

($psMTL_4$) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$, for every $x, y \in A$ (pseudo-prelinearity).

If A satisfies only the axioms $psMTL_1, psMTL_2$ and $psMTL_3$ then A is called a *pseudo residuated lattice*.

If additionally for any $x, y \in A$ the pseudo MTL-algebra A satisfies the axiom

($psMTL_5$) $(x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ (pseudo-divisibility), then A is a *pseudo BL- algebra*.

If A satisfies the axioms $psMTL_1, psMTL_2, psMTL_3$ and $psMTL_5$ then it is a *bounded divisible pseudo residuated lattice*. These structures were also studied under the name *bounded RL-monoids*.

A pseudo MTL- algebra A is called commutative if the operation $\odot$ is commutative. In this case the operations $\rightarrow$ and $\rightsquigarrow$ coincide, and thus, a commutative pseudo-MTL algebra is a MTL algebra.

In this chapter by A we denote the universe of a pseudo $MTL-$ algebra and by $C(A) = \{x \in A : x \odot (x \rightsquigarrow a) = (x \rightarrow a) \odot x$, for every $a \leq x, a \in A\}$. (Clearly, if A is a $MTL-$ algebra or a pseudo $BL-$ algebra, then $C(A) = A$.)

Also, we denote by $\mathcal{I}(A) = \{I \subseteq A :$ if $x, y \in A, x \leq y$ and $y \in I$, then $x \in I\}$ and by $\mathcal{I}'(A) = \{I = J \cap C(A), J \in \mathcal{I}(A)\}$.

So, in the case of pseudo $MTL-$ algebras we replace $\mathcal{I}(A)$ by $\mathcal{I}'(A)$.

If A is a $MTL-$ algebra or a pseudo $BL-$ algebra, then $\mathcal{I}'(A) = \mathcal{I}(A)$ is the set of all ordered ideals of A.

The original results of this chapter are contained in [**63**].

Chapter 4 : Open problems and new directions of research

This chapter contains some open problems relative to the topics of this thesis:

1. To characterize the MTL algebras A with the property that $Ds(A)$ is Stone lattice (respective, normal or co-normal lattice, MV algebra, LMn algebra).
2. The problem of the unicity (up to an isomorphism of Boolean algebras) for the maximal MTL (pseudo-MTL) algebra of quotients $Q(A)$ for a MTL (pseudo-MTL) algebra A.
3. A non-standard construction of the maximal MTL (pseudo-MTL) algebra of quotients as in the case of lattices (see [**91**]).
4. A study of localization without use the boolean center.
5. The translation of some properties of deductive systems (congruence filters) included in Chapter 1 in the case of ideals of a commutative ring.

and new directions of research:

1. *Development of a localization theory for residuated lattices (in commutative and non-commutative cases).*
2. *A study of pure filters and stable topology on on residuated lattices.*
3. *To obtain an orthogonal decompositions of prime filter spaces of residuated lattices.*
4. *To extinde for non-commutative case the results from* [**23**].
5. *A study of minimal pure filters in residuated lattices.* As model for this direction of research will be the papers [**72**], [**71**] and [**75**].
6. To study the structure of MTL algebra of localization for algebra Lindenbaum-Tarski of the logic MTL and the reflection of properties of logic MTL in this algebraic structure.
7. The study for a MTL algebra A of the different types of deductive systems (congruence filters)in $A_{\mathcal{F}}$ in connection with those of A.
8. The study of the position of the results of this thesis in connection with another domains as: topology, logic, informatics, etc.

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