Summary of the Ph. D. Thesis

"Unilateral problems in mathematical physics"

This thesis contains twelve chapters. The first chapter is devoted to the basic notation and results that are fundamental to the developments later in this thesis. We review the background on functional analysis that we need in the study of hemivariational inequalities, frictional contact problems or eigenvalue problems. The material presented is standard and can be found in many textbooks and monographs. For this reason, we do no present the proofs.

Except the first chapter, the following chapters are based on original research papers that are published or submitted to journals and are divided into three parts.

The first and the second part of this thesis are dedicated to the study of unilateral problems while the last part deals with the study of bilateral BVP's. A short and concise presentation of these parts follows.

- Part I **Hemivariational Inequalities** (chapters 2-6) is devoted to the mathematical study of several hemivariational inequalities. We present various classes of hemivariational inequalities for which we prove existence results and, for some of them, we prove uniqueness and stability results. With this end in view we use monotonicity, compactness, topological and fixed point arguments.
- Part II **Frictional Contact Problems** (chapters 7-9) focuses on the weak solvability of mechanical models describing the contact between a body and an obstacle called foundation. The envisaged processes are static and the materials are assumed to be elastic, but not necessarily linearly. We model the frictional contact with boundary conditions involving Clarke's generalized gradient of a functional which is not necessarily convex. Therefore, the variational formulations of the models lead to hemivariational or to variational-hemivariational inequalities. For each one of the problems, we provide a variational formulation then we establish existence and sometimes uniqueness results concerning the weak solutions.
- Part III **Nonlinear Eigenvalue Problems** (chapters 10-12) we focus our attention on the class of problems with degeneracies and singularities. Our goal is to prove the existence of weak solutions for certain partial differential equations (PDE's) of elliptic type involving the Laplace operator (chapters 10,11) or the forward difference operator (chapter 12).

Contents

Chapter 2 is based on the paper *Existence results for hemivariational inequalities involving relaxed* $\eta - \alpha$ *monotone mappings* published in *Communications in Applied Analysis.*

In this chapter we establish some existence results for hemivariational inequalities with relaxed $\eta - \alpha$ monotone mappings on bounded, closed and convex subsets in reflexive Banach spaces. Our proofs rely essentially on the Tarafdar Fixed Point Theorem for set valued mappings. We also provide a sufficient condition for the existence of solutions in the case of unbounded subsets.

Chapter 3 is based on the paper *On a class of variational-hemivariational inequalities involving set valued mappings* published in *Advances in Pure and Applied Mathematics.*

In this chapter, using the KKM Principle, we establish some existence results for variationalhemivariational inequalities involving monotone set valued mappings on bounded, closed and convex subsets in reflexive Banach spaces. We also derive several sufficient conditions for the existence of solutions in the case of unbounded subsets.

Chapter 4 is based on the paper *Hartman-Stampacchia results for stably pseudomonotone operators and nonlinear hemivariational inequalities* published in *Applicable Analysis* (2009 ISI Impact Factor: 0,613, rank 147/201 Applied Mathematics).

Here, we are concerned with two classes of nonstandard hemivariational inequalities. In the first case we establish a Hartman-Stampacchia type existence result in the framework of stably pseudomonotone operators. Next, we prove two existence results for a class of nonlinear perturbations of canonical hemivariational inequalities. Our analysis includes both the cases of compact sets and of closed convex sets in Banach spaces. Applications to noncoercive hemivariational and variational-hemivariational inequalities illustrate the abstract results of this chapter.

Chapter 5 is based on the paper *Nonlinear hemivariational inequalities and applications to nonsmooth Mechanics* published in *Advances in Nonlinear Variational Inequalities.*

The goal of this chapter is to establish several existence results for a class of nonstandard hemivariational inequalities. Our analysis includes both the cases of bounded and unbounded closed and convex subsets in real reflexive Banach spaces. The proofs strongly rely on the KKM Principle combined with the Mosco Alternative. In the last section of the chapter several applications illustrate the abstract results that were proved throughout the chapter.

Chapter 6 is based on the paper *Existence and uniqueness results for a class of quasi-hemivariational inequalities* published in *Journal of Mathematical Analysis and Applications* (2009 ISI Impact Factor: 1,225, rank 30/251 Mathematics).

In this chapter we are concerned with the study of a nonstandard quasi-hemivariational inequality. Using a fixed point theorem for set-valued mappings the existence of at least one solution in bounded closed and convex subsets is established. We also provide sufficient conditions for which our inequality possesses solutions in the case of unbounded sets. Finally, the uniqueness and the stability of the solution are analyzed in a particular case.

Chapter 7 is based on the paper *Weak solutions for nonlinear antiplane problems leading to hemivariational inequalities* published in *Nonlinear Analysis: Theory, Methods and Applications* (2009 ISI Impact Factor: 1,487, rank 18/251 Mathematics).

Throuhout this chapter we analyze the antiplane shear deformation of an elastic cylinder in frictional contact with a rigid foundation, for static processes, under the small deformations hypothesis. Based on the Knaster-Kuratowski-Mazurkiewicz Principle in the theory of the hemivariational inequalities, we prove that the model has at least one weak solution. Moreover, we present several examples of constitutive laws and friction laws for which our theoretical results are valid. Finally, we comment on the conditions which guarantee the uniqueness of solution.

Chapter 8 is based on the paper *Antiplane shear deformations of piezoelectric bodies in contact with a conductive support* submitted for publication to *Nonlinear Analysis: Theory, Methods and Applications.*

We consider a mathematical model which describes the frictional contact between a piezoelectric body and an electrically conductive support. We model the material's behavior with an electro-elastic constitutive law; the frictional contact is described with a boundary condition involving Clarke's generalized gradient and the electrical condition on the contact surface is modelled using the subdifferential of a proper, convex and lower semicontinuous function. We derive a variational formulation of the model and then, using a fixed point theorem for set valued mappings, we prove the existence of at least one weak solution. Finally, the uniqueness of the solution is discussed; the investigation is based on arguments in the theory of variationalhemivariational inequalities.

Chapter 9 is based on the paper *Contact models leading to variational-hemivariational inequalities* submitted for publication to *Zeitschrift für Angewandte Mathematik und Physik (ZAMP)* .

A frictional contact model, under the small deformations hypothesis, for static processes is considered. We model the behavior of the material by a constitutive law using the subdifferential of a proper, convex and lower semicontinuous function. The contact is described with a boundary condition involving Clarke's generalized gradient. Our study focuses on the weak solvability of the model. Based on a fixed point theorem for set valued mappings, we prove the existence of at least one weak solution. The uniqueness, the boundness and the stability of the weak solution are also discussed; the investigation is based on arguments in the theory of variational-hemivariational inequalities. Finally, we present several examples of constitutive laws and friction laws for which our theoretical results are valid.

Chapter 10 is based on the paper *Nonlinear, degenerate and singular eigenvalue problems on* $\mathbb{R}^N$ published in *Nonlinear Analysis: Theory, Methods and Applications* (2009 ISI Impact Factor: 1,487, rank 18/251 Mathematics).

The goal of this chapter is to establish the existence of a continuous family of eigenvalues lying in a neighborhood at the right of the origin for some nonlinear, nonhomogeneous, degenerate and singular elliptic operators on the whole space $\mathbb{R}^N$. Our proofs rely essentially on the Caffarelli-Kohn-Nirenberg inequality combined with the Banach fixed point theorem.

Chapter 11 is based on the paper *On an eigenvalue problem involving variable exponent growth conditions* published in *Nonlinear Analysis: Theory, Methods and Applications* (2009 ISI Impact Factor: 1,487, rank 18/251 Mathematics).

We study the problem $-\Delta u - \varepsilon \operatorname{div}\left((1+|\nabla u|^2)^{\frac{p(x)-2}{2}} \nabla u\right) = \lambda\,(u+\varepsilon)$ in Ω, $u=0$ on $\partial\Omega$, where Ω is a bounded domain in $\mathbb{R}^N$, $p : \bar{\Omega} \to (1,2)$ is a continuous function and λ and ε are two positive constants. We prove that for any $\varepsilon > 0$ each $\lambda \in (0, \lambda_1)$ is an eigenvalue of the above problem, where λ_1 is the principal eigenvalue of the Laplace operator on Ω. Moreover, for each eigenvalue $\lambda \in (0, \lambda_1)$ it corresponds a unique eigenfunction. The proofs will be based on the Banach fixed point theorem combined with adequate variational techniques.

Chapter 12 is based on the paper *Continuous family of eigenvalues concentrating in a small neighborhood at the right of the origin for a class of discrete boundary value problems* published in *Annals of the University of Craiova, Math. Comp. Sci. Ser.*.

In this chapter, we prove the existence of a continuous spectrum that lies in a neighborhood at the right of the origin for some nonlinear difference operators. Our proofs rely essentially on the Banach fixed point theorem and a minimization technique.