Ph. D. Thesis

Moment Problems and Applications in Control Theory

Summary

On this thesis we are interested to study the controllability properties of some partial differential equations (discrete and continuous) when a vanishing perturbation is added. More precisely, we want to find a control for the initial problem as limit of controls of the perturbed one. The extra term changes the type of our equation, usually from hyperbolic to parabolic, introducing a stronger dissipation in the system. Therefore, we can say that we are dealing with singular controllability problems. The addition of a viscous perturbative term is a common tool in the study of qualitative or numerical properties of dynamical systems. This procedure is useful, for instance, if we desire to regularize the solution or to filter out some unwanted spurious high oscillations. Our study shows that, in some particular but relevant situations, the controllability properties of the equation are not affected by the perturbation (in the continuous cases) or even improved by it (in the discrete cases). The method used, along this thesis, to study the controllability is a classical one which consists in reducing the controllability problem to a moment problem whose solutions are given in terms of an explicit biorthogonal sequence to a family of exponential functions.

We give now a short presentation of this method. Let $(\lambda_n)_n$ be a sequence of complex numbers (which are usually the eigenvalues of the unbounded differential operator corresponding to our problem). We reduce the control problem to a moment problem of the type: find a function $v \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ such that

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} v\left(t+\frac{T}{2}\right) e^{t\overline{\lambda}_n} dt = \widehat{a}_n e^{-\frac{T}{2}\overline{\lambda}_n} \qquad (\forall n), \tag{1}$$

where $(\widehat{a}_n)_n$ is a sequence of complex numbers (determined by the initial datum to be controlled). Once we have a biorthogonal sequence $(\theta_m)_m$ to the family of exponential functions $(e^{t\lambda_n})_n$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ a "formal" solution of (1) will be given by

$$v(t) = \sum_n \widehat{a}_n e^{-\frac{T}{2}\overline{\lambda}_n} \theta_n \left(t-\frac{T}{2}\right) \qquad (t \in (0,T)). \tag{2}$$

Now, the main task is to show that there exists a biorthogonal sequence $(\theta_m)_m$ and to evaluate its L^2-norm in order to prove the convergence of the series from the right hand side of (2), for each initial data we want to control.

Moreover, as we have said before, in all our problems the eigenvalues λ_n and the coefficients $\widehat{a}_n$ depend on one or two parameters which introduce the perturbation terms and are devoted to tend to zero. We are interested to study the controllability properties of our problems when those parameters vanish. If there exists a solution of problem (1) which is uniformly bounded with respect to our vanishing parameters we say that our problem is *uniformly controllable*. In this case we can pass to the limit and we can obtain a control of the unperturbed equation as limit of controls of the perturbed one.

The characteristic of our moment problems is the presence of values $(\lambda_n)_n \subset \mathbb{C}$ with vanishing real parts. While many results are known for purely imaginary or real values $(\lambda_n)_n$, fewer concern the complex case. Moreover, the fact that the real parts of λ_n tend to zero when our perturbation parameter vanishes makes the problem singular and more difficult.

This thesis contains six chapters and it begins with a short introduction. The first chapter contains a few general results that we have used on the main part of the thesis. The rest of the chapters are divided into two parts dedicated to the study of the controllability for some hyperbolic equations with vanishing viscosity and for some perturbed discrete problems, respectively.

Chapter 1, entitled **General results**, gives a briefly presentation of the basic results concerning the moment problems, the biorthogonal sequences and nonharmonic Fourier series. We mention here Müntz's Theorem, Ingham's Theorem, Szász's Theorem and Paley–Wiener Theorem. Moreover, we show how we can obtain a biorthogonal sequence when we are dealing with a family of real or purely imaginary exponential functions. These are known facts and we include them here just for comparison and for a better understanding of the method we use. In the last part of this chapter we give a brief description of the method we use to construct and evaluate biorthogonal sequences to general families of exponential functions. This construction was use for the first time by Paley and Wiener [18] in their study of completness properties of exponential families and, in a refined way, by Fattorini and Russell [10, 11] in the first proof of controllability of the heat equation.

Part I entitled *Controllability of some hyperbolic problems with vanishing viscosity* contains chapters 2–4. In this part we study the controllability properties of the wave, Schrödinger and beam equations when a viscous term is introduced. The viscous term depends on a small parameter and vanishes as the parameter goes to zero. This study is motivated by the fact that viscous terms are usually introduced in equations of hyperbolic type to increase the regularity of the solutions or to dissipate the high oscillations. Our aim is to show which of the controllability properties of the initial equation are preserved after the introduction of a viscosity and if we can obtain a limit control from controls of the perturbed system.

Chapter 2 entitled **A singular control problem for the wave equation** is based on the paper *A singular controllability problem with vanishing viscosity* written in collaboration with S. Micu (see [3]). In this chapter, we study the controllability properties of a vanishing viscosity approximation of the one dimensional wave equation and the relations with the ones of the conservative limit equation. The characteristic of the viscous term is that it contains the fractional power α of the Dirichlet Laplace operator. The parameter α belongs to $[0,1) \setminus \left\{\frac{1}{2}\right\}$ and through him we may increase or decrease the strength of the high frequencies damping which allows us to cover a larger class of dissipative mechanisms. The viscous term, being multiplied by a small parameter ε devoted to tend to zero, vanishes in the limit.

More precisely, we begin with the problem: given $T, \varepsilon > 0$ and $\alpha \in [0,1) \setminus \left\{\frac{1}{2}\right\}$ we consider the perturbed wave equation with "lumped" control

$$\begin{cases} u_{tt}(t,x) - \partial^2_{xx} u(t,x) + 2\varepsilon(-\partial^2_{xx})^{\alpha} u_t(t,x) + \varepsilon^2(-\partial^2_{xx})^{2\alpha} u(t,x) = v_\varepsilon(t) f(x) & (t,x) \in Q_T \\ u(t,0) = u(t,\pi) = 0 & t \in (0,T) \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x) & x \in (0,\pi), \end{cases} \tag{3}$$

where the profile $f \in L^2(0,\pi)$ is given such that all his the Fourier coefficients are different of zero and $Q_T = (0,T) \times (0,\pi)$.

Equation (3) is said to be null controllable in time $T > 0$ if, for any initial data $(u^0, u^1) \in \mathcal{H}_0 \subset H_0^1(0,\pi) \times L^2(0,\pi)$, there exists a control $v_h \in L^2(0,T)$ such that the corresponding solution verifies

$$u(T,x) = u_t(T,x) = 0 \qquad (x \in (0,\pi)). \tag{4}$$

The space $\mathcal{H}_0$ is defined as follows

$$\mathcal{H}_0 = \left\{ (u^0,u^1) \in H_0^1(0,\pi) \times L^2(0,\pi) \;\middle|\; \sum_{n\geq 1} \frac{n^2 \left|\widehat{u}_n^0\right|^2 + \left|\widehat{u}_n^1\right|^2}{\left|\widehat{f}_n\right|^2} < \infty \right\}, \tag{5}$$

where $(\widehat{u}_n^0)_{n\geq 1}$, $(\widehat{u}_n^1)_{n\geq 1}$ and $(\widehat{f}_n)_{n\geq 1}$ are the Fourier coefficients of the initial data (u^0,u^1) and of the profile f, respectively.

The term $2\varepsilon(-\partial_{xx}^2)^\alpha u_t(t,x)$ represents an added viscous term and $\varepsilon^2(-\partial_{xx}^2)^{2\alpha} u(t,x)$ allows us to consider a stronger dissipation, both are devoted to vanish as ε tends to zero. It is important to notice at this moment that if we only add the first viscous term, $2\varepsilon(-\partial_{xx}^2)^\alpha u_t(t,x)$, the resulting equation will remains dissipative but his controllability properties are poor. Indeed, the family of exponential functions corresponding to this case is given by $\Lambda = \left(e^{\nu_n t}\right)_{n\in\mathbb{Z}^*}$, where $\nu_n = \varepsilon|n|^{2\alpha} + \mathrm{sgn}\,(\mathrm{n})\sqrt{|n|^{4\alpha} - n^2}$. We notice that, if $\alpha > \frac{1}{2}$, we obtain

$$\lim_{n\to-\infty} \nu_n = 0,$$

which implies that the family Λ is not minimal. Consequently, the problem is not spectrally controllable if $\alpha > \frac{1}{2}$. Since the most interesting case corresponds to $\alpha > \frac{1}{2}$, it appears that stronger dissipation is needed. This is obtained by adding the second viscous term, $\varepsilon^2(-\partial_{xx}^2)^{2\alpha} u(t,x)$.

The null controllability problem is equivalent to find, for every initial data $(u^0,u^1) \in \mathcal{H}_0$, a solution $v_\varepsilon \in L^2(0,T)$ of the following moment problem:

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} v_\varepsilon\left(t + \frac{T}{2}\right) e^{\overline{\lambda}_n t} dt = -\frac{e^{-\overline{\lambda}_n \frac{T}{2}}}{\widehat{f}_{|n|}} \left(\widehat{u}_{|n|}^1 + \lambda_n \widehat{u}_{|n|}^0\right) \qquad (n \in \mathbb{Z}^*), \tag{6}$$

where $\lambda_n = in + \varepsilon|n|^{2\alpha}$ are the eigenvalues of the operator

$$\mathcal{W} = \begin{pmatrix} 0 & -I \\ -\partial_{xx}^2 + \varepsilon^2(-\partial_{xx}^2)^{2\alpha} & 2\varepsilon(-\partial_{xx}^2)^\alpha \end{pmatrix}$$

corresponding to the "adjoint" problem of (3).

We recall that $(\theta_m)_{m\in\mathbb{Z}^*} \subset L^2(-\frac{T}{2},\frac{T}{2})$ is a biorthogonal sequence to the family of exponential functions $\left(e^{\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ in $L^2(-\frac{T}{2},\frac{T}{2})$ if and only if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t) e^{\overline{\lambda}_n t} dt = \delta_{mn} \qquad (m,n \in \mathbb{Z}^*). \tag{7}$$

Once we have at our disposal a biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ to the family $\left(e^{\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ in $L^2(-\frac{T}{2},\frac{T}{2})$, we can give immediately a formal solution of (6) through the formula

$$v_\varepsilon(t) = -\sum_{m\in\mathbb{Z}^*} \frac{e^{-\overline{\lambda}_m \frac{T}{2}}}{\widehat{f}_{|m|}} \left(\widehat{u}_{|m|}^1 + \lambda_m \widehat{u}_{|m|}^0\right) \theta_m\left(t - \frac{T}{2}\right) \qquad (t \in (0,T)). \tag{8}$$

Now the main task is to show that there exists a biorthogonal sequence $(\theta_m)_{m\in\mathbb{Z}^*}$ to the family of exponential functions $\left(e^{\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ in $L^2(-\frac{T}{2},\frac{T}{2})$ and to evaluate its norm, in order to prove the convergence of the right hand side of (8) for any initial data $(u^0,u^1)\in\mathcal{H}_0$.

The main result of this chapter reads as follows:

Theorem 1. *Let $\alpha\in[0,1)\setminus\left\{\frac{1}{2}\right\}$ and $f\in L^2(0,\pi)$ be a function such that $\widehat{f}_n\neq 0$ for every $n\geq 1$. There exists a time $T>0$ with the property that, for any $(u^0,u^1)\in\mathcal{H}_0$ and $\varepsilon\in(0,1)$, there exists a control $v_\varepsilon\in L^2(0,T)$ of (3) such that the family $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is uniformly bounded in $L^2(0,T)$ and any weak limit v of it, as ε tends to zero, is a control in time T the wave equation.*

In order to justify the damping mechanism introduced in (3), which involves the fractional power α of the Laplace operator, let us point out that sometimes it may be useful to control the amount of dissipation introduced in the system not only by means of the vanishing parameter ε but also by an adequate choice of the differential operator. In (3) this is achieved through the parameter α. Note that, if $\alpha\in\left[0,\frac{1}{2}\right)$, the imaginary parts of the eigenvalues λ_n dominate the real ones and problem (3) has the same hyperbolic character as in the limit case $\varepsilon=0$. On the contrary, if $\alpha\in\left(\frac{1}{2},1\right)$, (3) has a parabolic type. In this case we are dealing with a truly singular control problem and the pass to the limit is sensibly more difficult. Finally, let us remark that $\alpha=\frac{1}{2}$ is a singular case in which the basic controllability properties (such as spectral controllability) of (3) do not hold.

For $\alpha\in[0,1)\setminus\{\frac{1}{2}\}$, the construction from the proof of the main result implies that the following Ingham's type inequality (see [15]) holds, for any finite sequence $(\beta_n)_{n\in\mathbb{Z}^*}$ and T sufficiently large,

$$C(T,\alpha)\sum_{n\in\mathbb{Z}^*}|\beta_n|^2e^{-\omega\varepsilon|n|^{2\alpha}}\leq\int_{-T}^{T}\left|\sum_{n\in\mathbb{Z}^*}\beta_n e^{\lambda_n t}\right|^2dt, \tag{9}$$

where $\varepsilon\in(0,1)$, ω is an absolute positive constant and C a positive constant depending of T and α but independent of ε. From this point of view this result extends the ones from [9, 12, 19], where Ingham's type inequalities are obtained under a more restrictive uniform sparsity condition of the sequence $(\lambda_n)_{n\in\mathbb{Z}^*}$. Indeed, one of the major difficulty in our study is related to the fact that the sequence of our eigenvalues $(\lambda_n)_{n\in\mathbb{Z}^*}$ are not included in a sector of the positive real axis and does not verify a uniform separation condition of the type

$$|\lambda_n-\lambda_m|\geq\delta|n^\beta-m^\beta|\qquad(n,m\in\mathbb{Z}^*),$$

for some $\beta>1$ and $\delta>0$ independent of ε. The fact that $C(T,\alpha)$ in (9) does not depend of ε is of fundamental importance since it ensures the uniform boundedness of a family of controls $(v_\varepsilon)_{\varepsilon\in(0,1)}$ for (3) and the possibility to pass to the limit in order to obtain a control v for the wave equation

$$\begin{cases} u_{tt}(t,x)-\partial^2_{xx}u(t,x)=v(t)f(x) & (t,x)\in Q_T\\ u(t,0)=u(t,\pi)=0 & t\in(0,T)\\ u(0,x)=u^0(x),\ u_t(0,x)=u^1(x) & x\in(0,\pi).\end{cases} \tag{10}$$

Chapter 3 entitled **Controllability of a Schrödinger equation** is based on the paper *Small time uniform controllability of the linear one dimensional Schrödinger equation with vanishing viscosity* written in collaboration with I. Rovenţa (see [4]). In this chapter, we discuss the linear one dimensional Schrödinger equation perturbed by a vanishing viscosity term depending on a small parameter ε. We are interested to prove that, for any time $T > 0$ and for each initial datum $u^0 \in H^{-1}(0,\pi)$, there exists an uniformly bounded family of boundary controls.

More precisely, we consider the following problem: for a given $T > 0$ and an initial datum $u^0 \in H^{-1}(0,\pi)$, we look for a boundary control of the Schrödinger equation as limit of the following perturbed problem

$$\begin{cases} u_t(t,x) - i\,u_{xx}(t,x) - \varepsilon u_{xx}(t,x) = 0 & x \in (0,\pi), \quad t > 0 \\ u(t,0) = 0, \ u(t,\pi) = v_\varepsilon(t) & t > 0 \\ u(0,x) = u^0(x) & x \in (0,\pi), \end{cases} \tag{11}$$

where $\varepsilon > 0$ is a small parameter devoted to tend to zero and $-\varepsilon u_{xx}$ represents the viscous term.

We emphasize that the focus of our concern is the uniform controllability with respect to ε of (11) and the possibility of obtaining controls for

$$\begin{cases} u_t(t,x) - i\,u_{xx}(t,x) = 0 & x \in (0,\pi), \quad t > 0 \\ u(t,0) = 0, \ u(t,\pi) = v(t) & t > 0 \\ u(0,x) = u^0(x) & x \in (0,\pi), \end{cases} \tag{12}$$

as limits, when ε goes to zero, of controls for (11). The interest of this problem is justified by the use of the vanishing viscosity as a typical mechanism to study Cauchy problems and to improve convergence of numerical schemes for hyperbolic conservation laws and shocks.

We reduce the control problem to a moment problem similar to (1) where the eigenvalues are given by $\lambda_n = \varepsilon n^2 - in^2$ for any $n \in \mathbb{N}^*$ and the coefficients $\widehat{a}_n = \frac{(-1)^n \pi}{2n(i+\varepsilon)} \widehat{u}_n^0$, where $(\widehat{u}_n^0)_{n\geq 1}$ are the Fourier coefficients of the initial datum u^0.

It is easy to see from the moment problem that, if $(\zeta_m)_{m\geq 1}$ is a biorthogonal sequence to the family of exponential functions $\left(e^{\lambda_n t}\right)_{n\geq 1}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$, then a control v_ε of (11) is given by

$$v_\varepsilon(t) = \sum_{n=1}^{\infty} \frac{(-1)^n \pi a_n}{2n(i+\varepsilon)} e^{-\frac{T}{2}\overline{\lambda}_n} \zeta_n\left(t - \frac{T}{2}\right) \qquad (t \in (0,T)), \tag{13}$$

provided that the right hand series converges in $L^2(0,T)$.

Now, the main task is to show that there exists a biorthogonal sequence $(\zeta_m)_{m\geq 1}$ and to evaluate its L^2-norm in order to prove the convergence of the series from (13) for each $u^0 \in H^{-1}(0,\pi)$. In order to do that, we introduce a family $\Psi_m(z)$ of entire functions of arbitrarily small exponential type (see, for instance, [20], page 61) such that $\Psi_m(i\overline{\lambda}_n) = \delta_{mn}$. Nextly, Paley–Wiener Theorem gives us a biorthogonal family $(\theta_m)_{m\geq 1}$ as the inverse Fourier transforms of $(\Psi_m)_{m\geq 1}$. Each Ψ_m is obtained from a Weierstrass product P_m multiplied by a function $M_{m,\varepsilon}$, called multiplier, with a suitable behavior on the real axis. Such a method was used for the first time by Paley and Wiener [18] and, in the context of control problems, by Fattorini and Russell [10, 11].

It is interesting to note that the simplest election for the function P_m does not work. Indeed, if we define P_m such that $P_m(i\lambda_n) = \delta_{mn}$, where λ_n are the eigenvalues of the operator corresponding to the adjoint problem of (11), we will be able to prove that P_m is an entire function with a good behavior on the real axis, but we will not succeed to construct an appropriate multiplier due to the lack of eigenvalues of our problem. To fix this problem we will add new values with the properties that between every two added values we have a gap γ and at most one eigenvalue of our problem.

The main result of this chapter reads as follows.

Theorem 2. *Let $T > 0$, $\varepsilon > 0$ and $u^0 \in H^{-1}(0, \pi)$. There exists a control $v_\varepsilon \in L^2(0, T)$ of* (11) *such that the family $(v_\varepsilon)_\varepsilon$ is uniformly bounded in $L^2(0, T)$ and any weak limit v of it is a control in time T for* (12).

Chapter 4 entitled **A note regarding the control of the beam equation** is based on the paper *Uniform controllability for the beam equation with vanishing structural damping* (see [6]). This chapter is devoted to study the effects of a vanishing structural damping on the controllability properties of the one dimensional linear beam equation.

More precisely, we study the possibility of obtaining a boundary control for the linear one dimensional beam equation as limit of controls of the following perturbed equation

$$\begin{cases} u_{tt}(t,x) + u_{xxxx}(t,x) - \varepsilon u_{txx}(t,x) = 0 & (t,x) \in Q_T \\ u(t,0) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0 & t \in (0,T) \\ u(t,\pi) = v_\varepsilon(t) & t \in (0,T) \\ u(0,x) = u^0(x) & x \in (0,\pi) \\ u_t(0,x) = u^1(x) & x \in (0,\pi), \end{cases} \tag{14}$$

where ε is a small parameter which tends to zero. We say that $v_\varepsilon \in L^2(0, T)$ is a control for (14) in time T if the corresponding solution verifies

$$u(T, x) = u_t(T, x) = 0 \qquad (x \in (0, \pi)), \tag{15}$$

for an initial data (u^0, u^1) in the space

$$\mathcal{H} = H^{-1}(0, \pi) \times V', \tag{16}$$

where

$$V = \left\{ \varphi \in H^3(0, \pi) \,|\, \varphi(0) = \varphi(\pi) = \varphi_{xx}(0) = \varphi_{xx}(\pi) = 0 \right\}.$$

If, for any initial data $(u^0, u^1) \in \mathcal{H}$, there exists a control $v_\varepsilon \in L^2(0, T)$ for (14) we say that the problem (14) is null controllable in time T. In (14), $-\varepsilon u_{txx}(t, x)$ represents the structural damping, devoted to vanish as ε goes to zero. The introduction of a vanishing term is a common tool in the study of Cauchy problems or in improving convergence of numerical schemes for hyperbolic conservation laws and shocks. Thus, a legitimate question is related to the behavior and the sensitivity of the controls during this process which is precisely the aim of this chapter.

The main result of this chapter is given by the following theorem.

Theorem 3. *There exists $T > 0$ with the property that, for any $(u^0, u^1) \in \mathcal{H}$ and $\varepsilon \in (0, 1)$, there exists a control $v_\varepsilon \in L^2(0, T)$ of* (14) *such that the family $(v_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^2(0, T)$ and any weak limit v of it is a control in time T of*

$$\begin{cases} u_{tt}(t,x) + u_{xxxx}(t,x) = 0 & (t,x) \in Q_T \\ u(t,0) = u_{xx}(t,0) = u_{xx}(t,\pi) = 0 & t \in (0,T) \\ u(t,\pi) = v(t) & t \in (0,T) \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x) & x \in (0,\pi). \end{cases} \tag{17}$$

Part II entitled *Controllability of some perturbed discrete problems* contains chapters 5 and 6. In this part we study the controllability properties of some discretizations corresponding to the heat and beam equations. It is well known that the good controllability properties of the continuous equations (wave, heat or beam) are lost when the usual finite differences schemes are considered. This phenomenon is due to the spurious high oscilations introduced by the discretization process (in the case of the hyperbolic type equations) or to the ill conditioning of the problem (in the case of the parabolic type equations). The aim of our study is to show that, under a perturbation which depends on the discretization parameter and vanishes in the limit, the approximation properties of the controls of the discrete systems are improved.

Chapter 5 entitled **An approximation the controls for the heat equation** is based on the paper *A numerical method with singular perturbation to approximate the controls of the heat equation* which is written in collaboration with S. Micu (see [2]). This chapter is devoted to analyze a numerical scheme for the approximation of the linear heat equation's controls. We consider a singular perturbation which consists of introducing the term $\varepsilon\partial_{tt}$ in the heat equation and transforming it into a wave equation. We manage to prove that there exists a sequence of controls of the corresponding perturbed semi-discrete systems which converges to a control of the original heat equation when both h (the mesh size) and ε (the perturbation parameter) tend to zero.

In this chapter we study the numerical implementation of the Hilbert Uniqueness Method for solving exact and approximate boundary controllability problems for the heat equation by reducing them to a minimization problem depending on the solutions of the adjoint equation. However, from the very beginning, the numerical experiments have shown that the efficient computing of the null controls for the heat equation is a difficult problem. The very weak coercivity of the functionals under consideration and the low regularity of the minimizers make the approximation problem exponentially ill-posed and the functional framework far from being well adapted to standard techniques in numerical analysis. For these reasons, how to construct robust numerical approximations of exact null controls for parabolic systems remains a challenging problem.

Since then, many alternative methods have been proposed and analyzed:

- [7] the controls of the heat equation are found by minimizing a cost functional with weights that blows up near the control time T;
- [16] a least squares type method is analyzed. Instead of working with solutions of the underlying state equation, and looking for one that may comply with the final desired state, they consider a suitable class of functions complying with required initial, boundary, and final conditions, and seek one of those that is a solution of the state equation;
- [17] a numerical version of the so–called transmutation method is developed.

In spite of this interesting studies, there are still relatively few results of convergence in the literature. We mention here the recent paper [1] in which special Carleman estimates are used to obtain uniform observability inequalities for semi–discrete semi–linear parabolic equations.

In this chapter we modify the heat equation with the aim to restore the stability of the corresponding backward system. More precisely, we consider $N \in \mathbb{N}^*$, a step $h = \frac{1}{N+1}$ and an equidistant mesh of the interval $(0, 1)$, $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ with $x_j = jh$ and $0 \leq j \leq N + 1$. Our perturbation technique and the classical central finite–difference approximation of the space derivates leads to the following semi–discretization of the one dimensional heat equation with boundary control

$$\left\{\begin{array}{lll} \varepsilon u_j''(t) + u_j'(t) - \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2} = 0 & 1 \le j \le N,\ t \in (0,T) \\ u_0(t) = 0 & t \in (0,T) \\ u_{N+1}(t) = v_h(t) & t \in (0,T) \\ u_j(0) = u_j^0 & 1 \le j \le N \\ u_j'(0) = u_j^1 & 1 \le j \le N, \end{array}\right. \tag{18}$$

where $\varepsilon = \varepsilon(h)$ and $\lim_{h\to 0} \varepsilon(h) = 0$. Hence, $\varepsilon u''$ represents a singular perturbation term which eventually vanishes in the limit as h tends to zero. Note that (18) is a second order differential system. As a consequence of this fact, a second initial data, u^1, is introduced.

The null controllability problem for (18) reads as follows: given any $T > 0$, $\varepsilon > 0$ and any initial data $(u_j^0, u_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$, there exists a control $v_h \in L^2(0,T)$ such that the corresponding solution $(u_j, u_j')_{1\le j\le N}$ of (18) verifies

$$u_j(T) = u_j'(T) = 0 \qquad (1 \le j \le N). \tag{19}$$

If the null controllability problem (18)−(19) has a solution $v_h \in L^2(0,T)$ for every initial data $U_h^0 = (u_j^0)_{1\le j\le N} \in \mathbb{C}^N$ and $U_h^1 = (u_j^1)_{1\le j\le N} \in \mathbb{C}^N$, we say that (18) is null controllable in time T.

In this case the moment problem has the eigenvalues given by

$$\lambda_n = \left\{\begin{array}{ll} \dfrac{1 + \operatorname{sgn}(n)\sqrt{1-4\varepsilon\mu_{|n|}}}{2\varepsilon} & \text{if} \quad \mu_{|n|} \le \dfrac{1}{4\varepsilon} \\ \\ \dfrac{1 + i\operatorname{sgn}(n)\sqrt{4\varepsilon\mu_{|n|}-1}}{2\varepsilon} & \text{if} \quad \mu_{|n|} > \dfrac{1}{4\varepsilon}, \end{array}\right. \qquad (1 \le |n| \le N), \tag{20}$$

and the coefficients $\widehat{a}_n = (-1)^n \frac{h}{\sin(|n|\pi h)} \left(\varepsilon a^1_{|n|h} - \varepsilon\lambda_n a^0_{|n|h} + a^0_{|n|h} \right)$, where $(a^0_{nh})_{n\ge1}$ and $(a^1_{nh})_{n\ge1}$ are the Fourier coefficients of the initial data (U_h^0, U_h^1) and

$$\mu_n = \frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right) \qquad (1 \le n \le N).$$

Theorem 4. *Let $T > 0$ and $u^0 \in L^2(0,1)$ such that*

$$u^0(x) = \sum_{n\ge1} a_n \sqrt{2}\sin(\pi n x). \tag{21}$$

There exist h_0, $c_0 \in (0,1)$ such that for any $h \in (0,h_0)$, $\varepsilon \in (0, c_0 h)$ and any initial data $(U_h^0, U_h^1) \in \mathbb{C}^{2N}$ of the form

$$(U_h^0, U_h^1) = \left(\sum_{j=1}^{N} a^0_{jh} \sqrt{2}\sin(njh\pi),\ 0 \right) \qquad (1 \le n \le N), \tag{22}$$

and with the property that

$$(a^0_{nh})_n \rightharpoonup (a_n)_n \quad \text{in} \quad \ell^2 \quad \text{when} \quad h \to 0, \tag{23}$$

there exists a family of exact controls $(v_h)_h \subset \mathcal{C}^1[0,T]$ for problem (18) which is uniformly bounded in $\mathcal{C}^1[0,T]$. Every limit function $v \in \mathcal{C}[0,T]$ of the family $(v_h)_h$ is a null control for

$$\begin{cases} u_t(t,x) - u_{xx}(t,x) = 0 & (t,x) \in (0,T)\times(0,1) \\ u(0,t) = 0 & t \in (0,T) \\ u(1,t) = v(t) & t \in (0,T) \\ u(x,0) = u^0(x) & x \in (0,1). \end{cases}$$

Finally, speaking about the HUM–controls, it is known that these are given by a solution of the homogeneous adjoint equation which minimizes a suitable quadratic cost. In order to obtain the minimizer we use the Conjugate Gradient Method and to solve the the differential equation that we encountered when we applied this method we use the Newmark Method.

Chapter 6 entitled **An approximation of the controls for the beam equation** is based on the paper *Approximation of the controls for the beam equation with vanishing viscosity* which is written in collaboration with S. Micu and I. Rovenţa [5]. In this chapter we consider a finite difference semi-discrete scheme for the approximation of the boundary exact controllability problem of the 1–D beam equation modelling the transversal vibrations of a beam with fixed ends.

$$\begin{cases} u_{tt}(t,x) + u_{xxxx}(t,x) = 0 & (t,x) \in (0,T)\times(0,1) \\ u(t,0) = u(t,1) = u_{xx}(t,0) = 0 & t \in (0,T) \\ u_{xx}(t,1) = v(t) & t \in (0,T) \\ u(0,x) = u^0(x) & x \in (0,1) \\ u_t(0,x) = u^1(x) & x \in (0,1). \end{cases} \tag{24}$$

It is known that, due to the high frequency numerical spurious oscillations, the uniform (with respect to the mesh-size) controllability property of the semi–discrete model fails in the natural setting. We then prove that, by adding a vanishing numerical viscosity, the uniform controllability property is restored.

We consider $N \in \mathbb{N}^*$, a step $h = \frac{1}{N+1}$ and a equidistant mesh of the interval $(0,1)$, $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ with $x_j = jh$ for any $j \in [0, N+1]$. Our perturbation technique and the central finite–difference approximation of the space derivates leads to the following semi–discretization of the beam equation.

$$\begin{cases} u_j''(t) + \frac{u_{j+2}(t)-4u_{j+1}(t)+6u_j(t)-4u_{j-1}(t)+u_{j-2}(t)}{h^4} - \varepsilon\frac{u_{j+1}'(t)-2u_j'(t)+u_{j-1}'(t)}{h^2} = 0 & 1 \le j \le N,\ t \in (0,T) \\ u_0(t) = 0, \quad u_{N+1}(t) = 0 & t \in (0,T) \\ u_{-1}(t) = -u_1(t), \quad u_{N+2}(t) = h^2 v_h(t) - u_N(t) & t \in (0,T) \\ u_j(0) = u_j^0(x), \quad u_j'(0) = u_j^1(x) & 1 \le j \le N, \end{cases} \tag{25}$$

where the parameter ε which multiplies the viscous term $\frac{u_{j+1}'(t)-2u_j'(t)+u_{j-1}'(t)}{h^2}$ depends on the mesh size h as follows

$$\lim_{h\to 0} \varepsilon(h) = 0.$$

We say that (25) is null controllable in time $T > 0$ if for any $\varepsilon > 0$ and any initial data $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1\le j\le N} \in \mathbb{C}^{2N}$ we find a control $v_h \in L^2(0,T)$ such that the corresponding solution $(u_j, u_j')_{1\le j\le N}$ of (25) verifies

$$u_j(T) = u_j'(T) = 0 \qquad (1 \le j \le N). \tag{26}$$

In this case the moment problem has the following particularities:

$$\lambda_n = \mu_n \, \frac{\varepsilon + i\,\sqrt{4-\varepsilon^2}}{2} \qquad (|n| \le N), \tag{27}$$

and

$$\widehat{a}_n = (-1)^n \frac{h}{\sqrt{2}\sin(|n|\pi h)} \left(-a^1_{|n|h} + \overline{\lambda}_n a^0_{|n|h} - \varepsilon \mu_{|n|} a^0_{|n|h} \right) \qquad (|n| \le N),$$

where $(a^0_{nh})_{n\ge 1}$ and $(a^1_{nh})_{n\ge 1}$ are the Fourier coefficients of the initial data (U^0_h, U^1_h) and

$$\mu_n = \frac{4}{h^2} \sin^2\left(\frac{n\pi h}{2}\right) \qquad (1 \le n \le N).$$

Taking into account that $|\lambda_n| = \mu_{|n|}$ which behave like n^2 we come across the same problems as the ones from the Schrödinger case. The difference in this case is made by the sine function from μ_n which divide the family of eigenvalues in two parts, one being finite and consists of the first and the last M values, while the second one being infinite and containing the rest of the eigenvalues. Moreover, thanks to the fact that the highest eigenvalues are found in the finite family, two biorthogonal sequences are needed, one for each family. If for the finite biorthogonal sequence the eigenvalues of our problem are sufficient for the infinite one we need to add new values with better gap properties (more precisely, with the same properties as the ones added in the Schrödinger case). Finally, the desire biorthogonal sequence will be obtain from the union of this two biorthogonals.

The main result of this chapter reads as follows.

Theorem 5. *Let $T > 0$ and $(u^0, u^1) \in \mathcal{H} = H^1_0(0,1) \times H^{-1}(0,1)$ with the Fourier expansions*

$$u^0(x) = \sum_{n\ge 1} \widehat{u}^0_n \sqrt{2}\sin(n\pi x) \qquad u^1(x) = \sum_{n\ge 1} \widehat{u}^1_n \sqrt{2}\sin(n\pi x). \tag{28}$$

There exists h_0, $c_0 > 0$ such that for any $h \in (0, h_0)$ and $\varepsilon \in \left(c_0 \frac{1}{T^2} h^2 \ln \frac{1}{h}, c_0 h\right)$ and any initial data $(U^0_h, U^1_h) \in \mathbb{C}^{2N}$ of the form

$$(U^0_h, U^1_h) = \left(\sum_{j=1}^{N} a^0_{jh} \sqrt{2}\sin(njh\pi), \sum_{j=1}^{N} a^1_{jh} \sqrt{2}\sin(njh\pi) \right) \qquad (1 \le n \le N), \tag{29}$$

and with the properties

$$(a^0_{nh})_n \rightharpoonup (\widehat{u}^0_n)_n \quad \text{in} \quad \ell^2 \quad \text{when} \quad h \to 0, \tag{30}$$

$$(a^1_{nh})_n \rightharpoonup (\widehat{u}^1_n)_n \quad \text{in} \quad \ell^2 \quad \text{when} \quad h \to 0, \tag{31}$$

there exists a family of exact controls $(v_h)_h \subset L^2(0,T)$ for problem (25) which converges to a null control for (24) in $L^2(0,T)$.

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