Differentiable Operators on Geometrical Structures

Ph. D. Thesis

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A central problem in differential geometry is relating algebraic properties of the Riemann curvature tensor to the geometry of the underlying manifold. As the curvature tensor R is in general difficult to deal with, we study several natural operators associated to R. Such examples are the Jacobi operator and other operators which are defined by the Jacobi operator, like the higher order Jacobi operator or the conformal Jacobi operator.

The Thesis contains an introduction, five chapters and the bibliography. The chapters are

I. Semi-Riemannian Osserman metric tensors on tangent and cotangent bundles of a Riemannian manifold

II. Generalized Osserman manifolds

III.Conformal Osserman manifolds

IV.Lorentz Osserman manifolds

V.Osserman condition on solvable Lie algebra of Iwasawa type.

In the first chapter we put forward some aspects of the geometry of the tangent bundle, following the monographies [8],[20] .If M is a differentiable manifold of dimension n, of class C^∞ , we successively construct the vertical and complete lifts on functions, 1-forms and then on tensorial fields.

In the above mentioned monographies, it is established that if M is a Riemannian manifold with the metric g, which has the local coordinates g_{ij} in the neighborhoods $U \subset M$, then

$$I \; g_{ij} dx^i \otimes dx^j \quad II \; 2g_{ij} dx^i \otimes \delta y^j \quad III \; g_{ij} \delta y^i \otimes \delta y^j$$

are quadratic forms globally defined on the tangent bundle TM over M. In more details, let $\pi : TM \to M$ be the canonical projection of TM on M. Then in the neighborhood $\pi^{-1}(U)$ of TM, $\delta y^h = dy^h + y^i \Gamma^h_{ij} dx^j$ represents the K.Yano and S. Ishihara‘s 1-forms, where $\Gamma^h_{ij}(x), x \in U$, are the Christoffel‘s symbols formed with g_{ij}. The metrics

$$
\begin{array}{ll}
II & 2g_{ij}dx^i \otimes \delta y^j \\
I+II & g_{ij}dx^i \otimes dx^j + 2g_{ij}dx^i \otimes \delta y^j \\
I+III & g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j \\
II+III & 2g_{ij}dx^i \otimes \delta y^j + g_{ij}\delta y^i \otimes \delta y^j
\end{array}
$$

are all nonsingular and can be defined as Riemannian or semi-Riemannian metrics on the tangent bundle TM over M. The metric II coincides with the complete lift g^C of the metric g on the tangent bundle and the metric I represents the vertical lift g^V of the metric g from the manifold to the tangent bundle.

We may remark that the metric

$$\overset{0}{G} = g_{ij}dx^i \otimes dx^j + 2g_{ij}dx^i \otimes \delta y^j + g_{ij}\delta y^i \otimes \delta y^j$$

i.e.I+II+III has a singular matrix $\begin{pmatrix} g_{ij} & g_{ij} \\ g_{ij} & g_{ij} \end{pmatrix}$.Consequently,$(TM, \overset{0}{G})$ cannot lead to more general theories then we have studied in the thesis.However, if we consider the metric

$$G = ag_{ij}dx^i \otimes dx^j + 2bg_{ij}dx^i \otimes \delta y^j + cg_{ij}\delta y^i \otimes \delta y^j$$

with $a, b, c \in \mathbb{R}$, and $ac - b^2 \neq 0$, then, according to a result due to M.Anastasiei, this is a semi-Riemannian metric on TM. The problem of geometrizing these metrics represents one of my future objectives.

Let (M, g) be a semi-Riemannian manifold and let

$$S_p^-(M) = \{Z \in T_pM \mid g(Z, Z) = -1\}$$

$$S_p^+(M) = \{Z \in T_pM \mid g(Z, Z) = 1\}$$

$$S_p(M) = \{Z \in T_pM \mid |g(Z, Z)| = 1\} = S_p^-(M) \cup S_p^+(M)$$

be the sets of unit timelike, spacelike and nonnull vectors on T_pM. Then

$$S^-(M) = \cup_{p \in M} S_p^-(M) = \{Z \in TM \mid g(Z, Z) = -1\}$$

$$S_+(M) = \cup_{p \in M} S_p^+(M) = \{Z \in TM \mid g(Z, Z) = 1\}$$

$$S(M) = \cup_{p \in M} S_p(M) = \{Z \in TM \mid |g(Z, Z)| = 1\}$$

are called the unit timelike, spacelike and nonnull bundle of (M, g).

In the following definitions, (M, g) is considered a semi-Riemannian manifold.

Definition 1.2.1 *If $Z \in S(M)$, then the operator $R_Z : Z^\perp \to Z^\perp$ defined by $R_Z X = R(X,Z)Z$ is called the Jacobi operator with respect to Z.*

Definition 1.2.2 *The manifold (M,g) is called timelike Osserman at $p \in M$ if the characteristic polynomial of R_Z is independent of $Z \in S_p^-(M)$. (M,g) is called spacelike Osserman at p if the characteristic polynomial of R_Z is independent of $Z \in S_p^+(M)$.*

In [3], p.4, is proved that (M,g) is timelike Osserman at p if and only if (M,g) is spacelike Osserman at p. This fact leads to the following:

Definition 1.2.3 . *(M,g) is called Osserman at p if (M,g) is both timelike and spacelike Osserman at p.*

Definition 1.2.5 . *The manifold (M,g) is called globally Osserman if the characteristic polynomial of R_Z is independent of $Z \in S^-(M)$ or $Z \in S^+(M)$.*

In the sequel, we study the metric I+II on the tangent bundle of a given Riemannian manifold (M,g). In the bellow theorem by the Osserman condition it is understood the globally Osserman condition.

Theorem 1.2.1[18] *Let (M,g) be a Riemannian manifold. Then the tangent bundle equipped with the metric I +II is a semi-Riemannian Osserman manifold with semi-defined metric tensor if and only if (M,g) is Osserman manifold with null eigenvalues of the Jacobi operator.*

Remark 1.2.4 *The result of the theorem* 1.2.1. *is still valid if we change the metric I+II on the tangent bundle with the deformed complet lift metric tensor $g_\Phi = g^C + \Phi^V$ of the Riemannian metric tensor g, where Φ is a symmetric $(0,2)-$tensor field on (M,g) and Φ^V denote the vertical lift of Φ.*

We will consider the $2n-$dimensional space $\mathbb{R}^{2n}$ with coordinates $(x,y) = (x^1,...,x^n,y^1,...,y^n)$,where $(x^1,...,x^n)$ are the usual coordinates on $\mathbb{R}^n$. Consider the deformed complet lift metric g_Φ,which is expressed by

$$g_\Phi = \sum_{i=1}^{n} dx^i \otimes dy^i + \sum_{i,j=1}^{n} \Phi_{ij}(x) dx^i \otimes dx^j$$

Of the six essential components of the curvature tensor on $(\mathbb{R}^{2n}, g_\Phi)$, the only nonvanishing component is

$$R(\partial_i^x, \partial_j^x)\partial_k^x = -\frac{1}{2}(\Phi_{il/j/k} + \Phi_{jk/i/l} - \Phi_{ik/j/l} - \Phi_{jl/i/k})\partial_l^y$$

where $\partial_i^x = \frac{\partial}{\partial x^i}, \partial_i^y = \frac{\partial}{\partial y^i}, \Phi_{ik/j/l} = \frac{\partial^2 \Phi_{ik}}{\partial x^j \partial x^l}$.

In the case in which $n = 2p$, we define on $\mathbb{R}^{2p}$ a complex structure by setting $J\partial_1^x = \partial_2^x, ..., J\partial_{2p-1}^x = \partial_{2p}^x$ and we consider J^C the complete lift of this structure on $\mathbb{R}^{2n} = T\mathbb{R}^n$.

Lemma 1.4.1.[17] *Let $X = \sum_{i=1}^{2p} (\alpha_i \partial_i^x + \alpha_{2p+i}\partial_i^y)$ be a vector field on $T\mathbb{R}^{2p}$. Then, the holomorphic sectional curvature with respect to every nondegenerate planes $\{X, J^C X\}$ is given by*

$$H(X) = -\frac{\sum\limits_{i,j,k,l=1}^{2p} \alpha_i\beta_j\alpha_k\beta_l(\Phi_{il/j/k} + \Phi_{jk/i/l} - \Phi_{ik/j/l} - \Phi_{jl/i/k})}{2(2\sum\limits_{i=1}^{2p} \alpha_i\alpha_{2p+i} + \sum\limits_{i,j=1}^{2p} \alpha_i\alpha_j\Phi_{ij})^2}$$

where $\beta_{2j-1} = -\alpha_{2j}, \beta_{2j} = \alpha_{2j-1}, j = 1, 2, ..., p$.

Corolar 1.4.1. Let Φ be an Hermitian symmetric $(0,2)-$tensor field on $(\mathbb{R}^2, g, J)$. Then $(\mathbb{R}^4, g_\Phi, J^C)$ is an Osserman semi-definite Kähler manifold and the holomorphic sectional curvature has the sign of $\Delta\Phi_{11}$, where Δ denotes the Laplacian on $\mathbb{R}^{\,2}$.

In the second chapter we will study the higher order Jacobi operator for some semi-Riemannan metrics on manifold $M = \mathbb{R}^{2p}$. The deformed complete metric lift of usual metric on $\mathbb{R}^p$, when its coefficients $\Phi_{ij} = \Phi_{ij}(x)$ depend only on x, is obtained.

The notion of generalized Osserman manifold in the context of Riemannian geometry is due to G. Stanilov and V. Videv in [13] and their results were extended to semi-Riemannian geometry in [7]. The Jacobi operator can be generalized as it follows. Let π be a nondegenerate k-plane in T_pM with orthonormal basis $\{e_1, ..., e_k\}$, where (M, g) is a Riemannian manifold of signature (r, s). The generalized Jacobi operator $J_R(\pi)$ is defined by

$$J_R(\pi) = \sum_{i=1}^{k} g(e_i, e_i)R(\cdot, e_i)e_i$$

The operator $J_R(\pi)$ is well defined (i.e. it is independent of the choice of orthonormal basis $\{e_1, ..., e_k\}$ and self-adjoint. We say that a pair of integers (r, s) is an admissible pair for T_pM if $0 \le r \le p$, $0 \le s \le q$ and $1 \le r + s \le p + q - 1$. This means that the Grassmannian $Gr_{(r,s)}(T_pM)$ of non-degenerate planes in T_pM of signature (r, s) is non-empty and does not consist of a single point.

Let (r, s) be an admissible pair. We say that (M, g) is Osserman of type (r, s) in $p \in M$ if the eigenvalues of the operator $J_R(\pi)$ do not depend on the choice of plane $\pi \in Gr_{(r,s)}(T_pM)$. P. Gilkey shows that if (M, g) is Osserman of type (r, s) then it is Osserman of type $(\widetilde{r}, \widetilde{s})$ for all admissible pairs $(\widetilde{r}, \widetilde{s})$ satisfying $r + s = \widetilde{r} + \widetilde{s}$ [6]. Thus, only the dimension $k = r + s$ of planes π is relevant and we simply talk about $k-$Osserman manifold. A semi-Riemannian manifold (M, g) is said to be a $k-$Osserman manifold if for all points $p \in M$, (M, g) is $k-$Osserman in p with the eigenvalues structure of $J_{R_p}(\cdot)$ independent of the chosen point p.

Let $M = \mathbb{R}^{2p}$ with coordinates $(x, y) = (x_1, ..., x_p, y_1, ..., y_p)$. Then $\mathcal{X} = Span_{1\le i\le p}\{\partial_i^x\}$, $\mathcal{Y} = Span_{1\le i\le p}\{\partial_i^y\}$ define two distributions of TM. The splitting $TM = \mathcal{X} \oplus \mathcal{Y}$ is just the usual splitting $T\mathbb{R}^{2p} = T\mathbb{R}^p \oplus T\mathbb{R}^p$. Let Φ be an $(0, 2)$ symmetric tensor field, where Φ_{ij} are $C^\infty-$functions. Then we define a semi-Riemannian metric of neutral signature (p, q) by setting

$$g_\Phi(x,y) = \sum_i dx^i \otimes dy^i + \sum_{i,j} \Phi_{ij}(x,y) dx^i \otimes dx^j \tag{2.1}$$

Proposition 2.2.1. *The components of the curvature tensor of the metric g_Φ on M are given by*

$R(\partial_i^x, \partial_j^x)\partial_k^x = (\frac{1}{2}\Phi_{ik}\ |_{l|j} - \frac{1}{2}\Phi_{jk}\ |_{l|i} + \frac{1}{4}\Phi_{jk}\ |_s\ \Phi_{is}\ |_l - \frac{1}{4}\Phi_{ik}\ |_s\ \Phi_{js}\ |_l)\partial_l^x +$
$+[\frac{1}{4}\Phi_{jk}\ |_s\ (\Phi_{sd|i} + \Phi_{is|d} - \Phi_{di|s}) + \frac{1}{4}\Phi_{id}\ |_l\ (\Phi_{jl|k} + \Phi_{lk|j} - \Phi_{kj|l}) -$
$-\frac{1}{4}\Phi_{ik}\ |_s\ (\Phi_{sd|j} + \Phi_{js|d} - \Phi_{dj|s}) - \frac{1}{4}\Phi_{jd}\ |_l\ (\Phi_{il|k} + \Phi_{lk|i} - \Phi_{ki|l}) -$
$-\frac{1}{4}\Phi_{jk}\ |_s\ \Phi_{is}\ |_l\ \Phi_{ld} + \frac{1}{4}\Phi_{ik}\ |_s\ \Phi_{js}\ |_l\ \Phi_{ld} - \frac{1}{4}\Phi_{jk}\ |_s\ \Phi_{id}\ |_l\ \Phi_{sl} +$
$+\frac{1}{4}\Phi_{ik}\ |_s\ \Phi_{jd}\ |_l\ \Phi_{sl} + (\frac{1}{2}\Phi_{jk}\ |_{s|i} - \frac{1}{2}\Phi_{ik}\ |_{s|j})\Phi_{sd} +$
$+\frac{1}{2}(\Phi_{jd|k|i} - \Phi_{kj|d|i} - \Phi_{id|k|j} + \Phi_{ki|d|j})]\partial_d^y.$
$R(\partial_i^x, \partial_j^x)\partial_a^y = [\frac{1}{2}(\Phi_{jl}\ |_{a|i} - \frac{1}{2}\Phi_{il}\ |_{a|j}) + \frac{1}{4}(\Phi_{js}\ |_a\ \Phi_{il}\ |_s - \Phi_{is}\ |_a\ \Phi_{jl}\ |_s)]\partial_l^y$
$R(\partial_i^x, \partial_a^y)\partial_b^y = -\frac{1}{2}\Phi_{id}\ |_b|_a\ \partial_d^y$ (2.6)
$R(\partial_a^y, \partial_j^x)\partial_k^x = [\frac{1}{4}\Phi_{jk}\ |_s\ \Phi_{sl}\ |_a + \frac{1}{2}\Phi_{jk}\ |_s|_a\ \Phi_{sl} - \frac{1}{4}\Phi_{ks}\ |_a\ \Phi_{jl}\ |_s +$
$+\frac{1}{2}(\Phi_{jl|k}\ |_a - \Phi_{kj|l}\ |_a)]\partial_l^y - \frac{1}{2}\Phi_{jk}\ |_s|_a\ \partial_s^x$
$R(\partial_a^y, \partial_b^y)\partial_k^x = R(\partial_a^y, \partial_b^y)\partial_c^y = 0,$
where $\Phi_{ij|k} = \partial_k^x\Phi_{ij}, \Phi_{ij}\ |_a = \partial_a^y\Phi_{ij}$.

Theorem 2.2.1. *We can choose the functions $\Phi_{ij}=\Phi_{ij}(x,y)$ such that the only nonvanishing component of the curvature tensor in the proposition* 2.2.1. *is*

$$R(\partial_i^x, \partial_j^x)\partial_k^x = C_{kji}^l \partial_l^y \tag{2.7}$$

We obtain in this way metrics g_Φ of the (2.1)-type, where the the functions Φ_{ij} are given by

$$\Phi_{ij}(x,y) = \begin{cases} f_{ii}(x) + y^i g_i(x^i), & \text{if } i = j \\ f_{ij}(x), & \text{if } i \neq j \end{cases} \tag{2.11}$$

Theorem 2.2.3. *If $p \geq 2$ then (M, g_Φ) is k Osserman for every admissible k, in the case in which Φ_{ij} are given by (*2.11*).*

In the work [4] the authors consider a family of metrics of the form

$$\begin{aligned} g_{(f_1,f_2)} &= y^1 f_1(x^1,x^2)dx^1 \otimes dx^1 + y^2 f_2(x^1,x^2)dx^2 \otimes dx^2 + \\ &+a(dx^1 \otimes dx^2 + dx^2 \otimes dx^1) + \\ &+b(dx^1 \otimes dx^3 + dx^3 \otimes dx^1 + dx^2 \otimes dx^4 + dx^4 \otimes dx^2) \end{aligned} \tag{2.16}$$

where $a,b \in \mathbb{R}$ and $f_1, f_2$are smooth real valued functions satisfying the condition

$$\frac{\partial f_1}{\partial x^2} = \frac{\partial f_2}{\partial x^1} \tag{2.17}$$

In the particular case in which $f_1(x^1,x^2) = f_1(x^1)$ and $f_2(x^1,x^2) = f_2(x^2)$, the condition (2.17) is trivially verified and we are lead to either locally symmetric or non-locally symmetric semi-Riemannian manifolds, depending on the conditions 1)-4) in the theorem 3 of [4].These metrics are of (2.1.)- type with the conditon (2.11). In this way, we obtain a new class of spaces, which we have called *locally symmetric generalized Osserman spaces, with the 2-nilpotent generalized Jacobi operator.*

In the third chapter we construct several examples of semi-Riemannian manifolds, which are Osserman in a given point. Let V be a vector space. These examples are essential because they determine the family of curvaturelike tensors constructed by P.Gilkey by using Clifford modules structures. A quadrilinear map $\mathcal{F} : V \times V \times V \times V \to \mathbb{R}$ is called curvaturelike if it satisfies the symmetries of the curvature tensor and the first Bianchi identity. Let $\langle .,. \rangle$ be a scalar product on the vector space V. We say that $F : V \times V \times V \to V$ is called curvaturelike trilinear map on V if the quadrilinear map $\mathcal{F}$ defined by $\mathcal{F}(X,Y,Z,W) = \langle F(X,Y)Z, W \rangle$ is curvaturelike for every X,Y,Z,W in V.

Let (M,g) be a Riemannian manifold of dimmension m, $\nabla-$ the Levi-Civita connection. The curvature tensor $\mathcal{R}$ is defined by setting

$$\mathcal{R}(X,Y,Z,W) = g(R(X,Y)Z,W) = g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)$$

where R is the curvature operator of (M,g).

If $\{e_i\}$ is a local orthonormal frame for the tangent bundle, the Ricci tensor and the scalar curvature are given by

$$\rho(e_i, e_j) = \sum_k \mathcal{R}(e_i, e_k, e_k, e_j)$$

respectively $\tau = \sum_i \rho(e_i, e_i)$. The Ricci operator is defined by setting

$$\rho(e_i) = \sum_j \rho(e_i, e_j) e_j.$$

We introduce the tensors L and R_0 by setting

$$L(X,Y)Z = g(\rho Y, Z)X - g(\rho X, Z)Y + g(Y,Z)\rho X - g(X,Z)\rho Y$$

$$R_0(X,Y)Z = g(Y,Z)X - g(X,Z)Y$$

In the following, we study curvaturelike operators of the type

$$V(X,Y)Z = R(X,Y)Z + \alpha\tau R_0(X,Y)Z + \beta L(X,Y)Z, \tag{3.2}$$

where α and β are constants. Let Φ be an almost complex Hermitian structure on TM $(m = 2n)$ and

$$R_\Phi(X,Y)Z = g(\Phi Y, Z)\Phi X - g(\Phi X, Z)\Phi Y - 2g(\Phi X, Y)\Phi Z.$$

We say that (M,g) is a complex space form if $R = \lambda_0 R_0 + \lambda R_\Phi$, for smooth functions λ_0 and λ, where $\lambda \neq 0$.

Theorem3.2.1. *Let us take* (M, g, Φ), *where we assume that* $\dim M = m \geq 8$, *which satisfies* $V = \lambda_0 R_0 + \lambda R_\Phi$, $\lambda \neq 0$, V *being defined by (3.2.) with* $\beta = -\frac{1}{m-2}$. *If* $\{X, \Phi X, Y, \Phi Y, Z, \Phi Z\}$ *is an orthonormal set, then*

1) $\Phi\Phi_{;X} = -\Phi_{;X}\Phi$.

2) $\Phi_{;X}$ *and* $\Phi\Phi_{;X}$ *are skew-adjoint.*

3) $g(\Phi_{;Z}Y - \Phi_{;Y}Z, X) = 0$.

4) $\Phi_{;X}X = 0$ *;* $\Phi_{;X}\Phi X = 0$.

5) $\Phi_{;X}Y + \Phi_{;Y}X = 0$.

6) $\nabla\Phi = 0$.

7) (M, g, Φ) *is a complex space form.*

By $\Phi_{;X}$ *we noted the covariant derivative of* Φ *with respect to* X.

Remark 3.2.1 *If we choose* $\alpha = \frac{1}{(m-1)(m-2)}$ *in* (3.2), *then* V *becames the Weyl curvature operator.From here it is realised how important are the considered operators. So, we are lead to the following*

Definition 3.2.1 *The conformal Jacobi operator is the Jacobi operator associated to the Weyl curvature operator, i.e.*

$$J_W(X)(Y) = W(Y, X)X$$

We say that (M, g) *is conformally Osserman if the eigenvalues of* J_W *are constant on the set of unit vectors in* T_pM.

Finally, we obtain a characterization of the conformal Osserman manifolds in the sense due to Chi[1] for Riemannian Osserman manifolds.

In the fourth chapter, we study Lorentzian manifolds which are semi-Riemannian manifolds of index 1. This chapter make the subject of the work [19]. There are certain differences between timelike vectors and spacelike vectors. An essential difference is that the orthogonal space to a timelike vector has definite induced inner product, unlike to a spacelike vector which has semi-definite induced inner product on its orthogonal space.

In the case of a Lorentzian manifold M, the following conditions are equivalent :

a) M is spacelike Osserman in $p \in M$.

b) M is timelike Osserman in $p \in M$.

c) M has constant sectional curvature at a point $p \in M$.

Theorem 4.1.2. *Let* M *be the Lorentzian manifold* $M = N \times \mathbb{R}$, *with the product metric* $g = g_N - dt^2$, *of* $\dim M \geq 4$, *where* (N, g_N) *is a Riemannian manifold. If* $\|W\|_g = 0$, *then* M *is spacelike Lorentz Osserman manifold, respectively timelike Lorentz Osserman manifold.*

Although the orthogonal space to a null vector is degenerate, by quotienting out its degenerate part, we obtain a definite inner product quotient space. This leads us to define the Jacobi operator with respect to a null vector. Next, using the concept of null isotropy, defined in [2], we have

Remark 4.1.1 *The semi-definite semi-Riemannian manifold* M *of dimension* $m \geq 3$ *is null isotropic at* $p \in M$ *if and only if the Weyl curvature tensor* $W = 0$ *at* $p \in M$.

Lemma 4.1.3. *Let M_1 and M_2 be semi-Riemannian manifolds with $M = M_1 \times M_2$ a semi-definite semi-Riemannian manifold and* $\dim M \geq 3$. *Then M is null isotropic if* $\dim M_1 = 1, \dim M_2 \geq 3$ *and M_2 is of constant curvature.*

The following remark is a consequence of the previous remark 4.1.1. and the lemma 4.1.3.

Remark 4.1.2. *Because in $M = N \times \mathbb{R}$ we have* $\dim \mathbb{R} = 1$ *and, according to theorem* 4.1.2, *N has a constant curvature, it follows that the considered Weyl's tensor in the same theorem* 4.1.2. *vanishes.*

In order for us to apply the theories exposed in the previous chapters, we analize in the chapter V the Osserman condition on solvable Lie groups of Iwasawa type.

Definition 5.1.5. A solvable Lie algebra s with inner product $\langle ., . \rangle$ is a metric Lie algebra of Iwasawa type if it satisfies the conditions

1) $s = a \oplus n$, where $n = [s, s]$ and the complement orthogonal of n, a, is abelian.

2) All operators ad_H, $H \in a$, are symmetric.

3) For some $H_0 \in a$, $ad_{H_0} \mid_a$has positive eigenvalues.

Let M be a Riemannian manifold, $\mathcal{R}$ be its curvature tensor and R_X be the associated Jacobi operator. If R_X has constant eingenvalues, independently on $X \in T_pM$ and $p \in M$, then we say that the manifold satisfies the Osserman condition. This condition is satisfied by rank 1 symmetric spaces, because they are 2-homogeneous, hense the group of the local isometries acts transitively on the bundle of the unit sphere.

The simply connected Lie group S, with the Lie algebra s, and the left invariant metric induced by the scalar product $\langle \cdot, \cdot \rangle$ will be called of Iwasawa type. If s is a Lie algebra of Iwasawa type, which satisfies the Osserman condition, then S has a constant negative sectional curvature and $\dim a = 1$. Whenever $\dim a = 1$, we say that the Lie algebra is of rank 1. We note by z the center of n and by v its orthononal complement relative to the metric restricted to n. Using the adjoint representation ad_H, where $H \in a$, and the map $j_Z : v \to v$ defined by $j_Z X = (ad_X)^* Z$, where $(ad_X)^*$ is the adjoint operator of ad_X, we obtain the formulas of the Jacobi operators on the Lie groups of Iwasawa type.Then we characterize the simply connected Lie group S, with the left invariant metric associated to a Lie algebra of Iwasawa type s, for which $n = [s, s]$ is 2-nilpotent, and satisfies the Osserman conditions. Finally, we have studied the generalized Jacobi operator on solvable Lie algebras of Iwasawa type.

References

[1] Q.S.Chi, Acurvature characterization of certain locally rank one symmetric spaces, J.Diff.Geom. 28(1998),187-202

[2] E.Garcia-Rio, D.Kupeli, Null and infinitesimal isotropy in semi-Riemannian geometry, J.Geom.Phys. 13(1994) 207-222

[3] E.Garcia-Rio, D.Kupeli, R.Vazquez-Lorenzo, Osserman manifolds in semi-Riemannian geometry, (Lecture Notes in Mathematics) Springer 2001

[4] E.Garcia-Rio, M.E.Vazquez-Abal, R.Vazquez-Lorenzo, Nonsymmetric Osserman pseudo-Riemannian manifolds, Proc. Amer. Math. Soc., 126(1998), 2771-2778

[5] P.Gilkey, Generalized Osserman Manifolds, Abh. Math. Sem. Univ. Hamburg, 68 (1998),125-127

[6] P.Gilkey, Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor, World Scientific Press, 2001

[7] P.Gilkey, G.Stanilov, V.Videv, Pseudo-Riemannian Manifolds whose Generalized Jacobi Operator has Constant Characteristic Polynomial, J.Geom., 62 (1998), 144-153

[8] R.Miron, M.Anastasiei, Fibrate vectoriale. Spatii Lagrange. Aplicatii in teoria relativitatii, Ed.Academiei, Bucuresti, 1987

[9] R.Miron, M.Anastasiei, The geometry of Lagrange spaces. Theory and Applications, Kluwer Acad. Publ.,1994

[10] R.Miron, D.Hrimiuc, H.Shimada,V.S.Sabau, The geometry of Hamilton and Lagrange spaces, Kluwer Acad. Publ.,2000

[11] B.O'Neill, Semi-Riemannian Geometry with applications to relativity, Academic Press, New York, 1983

[12] R.Osserman, Curvature in the eighties, Amer. Math. Monthly, 97 (1990), 731-756

[13] G.Stanilov, V.Videv, Four dimensional pointwise Osserman manifolds, Abh. Math. Sem. Univ. Hamburg, 68 (1998),1-6

[14] P.Stavre, Capitole speciale de geometrie diferentiala.Varietati pseudo-Riemanniene, Ed.Radical, Craiova, 2001

[15] P.Stavre, Aprofundari in geometria diferentiala, vol.1, E.U.C., 2004

[16] P.Stavre, Fibrate vectoriale, vol.1, E.U.C., 2004

[17] C.Sterbeti, About a Higher Order Osserman Manifolds, Proc. Non-Euclidean Geometry in Modern Physics (2002), 31-34

[18] C.Sterbeti, About a Semi-Riemannian Osserman Manifolds, The fifth Congress of the Romanian Mathematicians, Pitesti, 22-28 June 2003

[19] C.Sterbeti, Lorentz Osseman Manifolds, The National Conference of Hamilton-Lagrange spaces, Brasov, 22-25 September 2004(in preparation)

[20] K.Yano, S.Ishihara, Tangent and cotangent bundles, Marcel Dekker, New York, 1973