Universitatea din Craiova,
Facultatea de Matematică-Informatică

Simona Maria DĂBULEANU

Nonlinear analysis methods for the study of a class of evolution problems

Ph.D. Thesis

-abstract-

Supervisor:
Prof. Univ. Dr. Vicenţiu RĂDULESCU

Craiova - January 2005

1. Introduction

I. **Physical origin of the equation.** In this thesis we study viscous Hamilton-Jacobi equations of the form:

$$(VHJ) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = a|\nabla u|^p & \text{in} \quad (0,+\infty)\times\Omega, \\ u(0) = \mu_0 & \text{in} \quad \Omega. \end{cases}$$

with homogeneous Dirichlet boundary condition :

$$(D) \qquad u = 0 \quad \text{on} \quad (0,+\infty)\times\partial\Omega,$$

or homogeneous Neumann boundary condition:

$$(N) \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad (0,+\infty)\times\partial\Omega.$$

The domain $\Omega \subset \mathbb{R}^N$ is an open and bounded set with smooth boundary $\partial\Omega$. For $x \in \partial\Omega$, we denote by $\nu(x)$ the unit outward to the boundary $\partial\Omega$ in the point x. The parameters a, p will be chosen such that $a \in \mathbb{R}, a \neq 0$, $p > 0$ and the initial data μ_0 is a bounded Radon measure on Ω, or a measurable function in the Lebesgue space $L^q(\Omega)$, $q \geq 1$, or a continuous function on $\overline{\Omega}$.

The equations (VHJ) possess both mathematical and physical interest. Indeed, it is the simplest example of parabolic PDE with a nonlinearity depending on the first order spatial derivatives of the solution. On the other hand these equations appears in the stochastic and deterministic control theory as well as in the large deviations theory (see [3, 13, 14, 16]). More precisely if we denote by $P_\varepsilon(t,x)$ the probability that a diffusion process starting from x at time t with a small noise of $\varepsilon > 0$ intensity, remains in the bounded domain $\Omega \in \mathbb{R}^N$ until the fixed time $T > 0$, then $P_\varepsilon(t,x)$ satisfies a linear parabolic equation in $[0,T]\times\Omega$ with boundary conditions:

$$P_\varepsilon(t,x) = 0, \quad \text{for} \quad (t,x) \in [0,T]\times\partial\Omega \quad \text{and} \quad P_\varepsilon(T,x) = 1 \quad \text{for} \quad x \in \Omega.$$

The large deviation theory deals with the behavior of P_ε in the neighborhood of 0 as ε goes to 0. A classical method which may answer to this question is to consider the function u_ε defined by:

$$u_\varepsilon = -\varepsilon \log P_\varepsilon.$$

If the diffusion process is the Brownian motion with $(2\varepsilon^{\frac{1}{2}})$ intensity, then the function u_ε satisfies the equation:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \varepsilon\Delta u_\varepsilon = |\nabla u_\varepsilon|^2 & \text{in} \quad (0,+\infty)\times\Omega, \\ u_\varepsilon = +\infty & \text{on} \quad (0,+\infty)\times\partial\Omega, \\ u_\varepsilon(T) = 0 & \text{in} \quad \Omega, \end{cases}$$

This model corresponds to (VHJ) problem with Dirichlet boundary condition. For the Neumann condition we may consider diffusion processes with reflexive trajectories on the boundary Ω.

We mention that these equations appear also in the physical theory of growing surfaces due to a variety of mechanism. One such a mechanism is Ballistic deposition. A simple-minded picture of this mechanism is that of particles each moving along a straight path approaching a surface and randomly attaching themselves. This point of view is considered appropriate for vapor deposition and the sputter deposition of thin films of aluminium and rare earth metals. The accepted starting point for a continuum approach to this mechanism is the partial differential equation:

$$(KPZ_1) \qquad \frac{\partial h}{\partial t} = \gamma \Delta h + \lambda |\nabla h|^2 + \eta.$$

With this model, surface growth relative to a reference plane, which may move with a constant velocity, is simulated. The unknown h denotes the height of the surface above the plane and t denotes the time. The first term on the right-hand side of equation (KPZ_1) describe a diffusive relaxation in which γ may be thought of as an effective surface tension. The second term on the right-hand side arises from the growth process. The constant λ in this term is a measure of the net rate of deposition. Finally, the last term on the right-hand side is produced by a stochastic force with zero mean and short-range correlation. In its simplest form η signifies the white noise with Gaussian distribution. The above model was first proposed by Kardar, Parisi and Zhang [19], and has since been referred as KPZ equation.
Taking into account further surface growth effects J.Krug-H.Spohn, [20] extended the model (KPZ_1) to:

$$(KPZ_2) \qquad \frac{\partial h}{\partial t} = \gamma \Delta h + \lambda |\nabla h|^p + \eta.$$

With any value $p > 0$, (KPZ_2) has designated the generalized KPZ equation. In particular, without the noise term η it is known as the generalized deterministic KPZ equation.

II. **Some References.** In this paragraph we summarize the main contributions done in the study of equation of (VHJ) type or related problems to this equation.
In the whole space $\mathbb{R}^N$, the Cauchy problem has been intensively studied. L.Amour-M.Ben-Artzi [2] and M. Ben-Artzi [4] had proved the global in time existence of classical solutions for the Cauchy problem (VHJ) as long as $p \geq 1$ and the initial data $\mu_0 \in C^2(\mathbb{R}^N)$ is bounded together with her first and second derivatives. Later on B.Gilding-M.Guedda-R.Kersner [18] extended these results to $\mu_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $p > 0$.
If the initial data μ_0 is not a smooth one, more exactly if $\mu_0 \in L^q(\mathbb{R}^N)$, $1 \leq q < \infty$ or is a bounded Radon measure, M.Ben-Artzi-Ph. Souplet-F.B.Weissler in [6] and S.Benachour-Ph.Laurençot in [10] established some existence and non-existence results of weak solution for problem (VHJ) depending on parameter $a \in \mathbb{R}, a \neq 0$ and the exponent $p \geq 1$. Beside the existence and uniqueness of solutions for the Cauchy problem, S. Benachour-Ph. Laurençot-D. Schmitt in [8], S. Benachour-Ph. Laurençot-D. Schmitt-Ph. Souplet in [9], B. Gilding-M. Guedda-R. Kersner in [18], M.Ben-Artzi-H.Koch in [5] and Ph.Laurençot-Ph.Souplet in [21] studied the long time behavior of these solutions.
In bounded domains $\Omega \subset \mathbb{R}^N$, the problem (VHJ) with Dirichlet boundary condition (D) or Neumann boundary condition (N) has been little investigated.
We can mention here the paper of N. Alaa [1], who proved the existence and uniqueness of mild solutions when the initial data is a bounded Radon measure and $p \in [1, \frac{N+2}{N+1})$. Another

important contribution is the work of M.G.Crandall-P.L.Lions-P.E.Souganidis [11]. Using the theory of order preserving semigroup they proved the existence of a universal bound for the positives solutions of the Cauchy-Dirichlet problem as $p > 1$, $a < 0$ and μ_0 a positive and continuous function which vanishes on the boundary $\partial\Omega$ (Theorem 2.1, p.172 in [11]). Finally, we cite a recent result of Ph. Souplet [24], who proved the blow-up of the gradient of solutions for $a > 0$, $p > 2$ and initial data "large enough".

2. Main Results

This work continue the study of the Cauchy-Dirichlet problem $[(VHJ) + (D)]$ and Cauchy-Neumann problem $[(VHJ) + (N)]$, trying to clarify questions like existence of weak solutions, uniqueness, regularity and long time behavior of solutions as t goes to $+\infty$. A particular interest is given to problems $[(VHJ) + (D)]$ and $[(VHJ) + (N)]$ for irregular initial data μ_0. In the sequel $\mathcal{M}_b(\Omega)$ is the space of bounded Radon measures. We denote by $(e^{t\Delta})_{t\leq 0}$ and $(S(t))_{t\geq 0}$ the semigroup of contraction in $L^q(\Omega), q \geq 1$ related to the heat equation with homogeneous Dirichlet or Neumann boundary condition (see [23]). As we can see in [7] and [12] this semigroup can be extended, in a natural way, to the space of bounded Radon measures, $\mathcal{M}_b(\Omega)$.

I. **Cauchy-Dirichlet problem for viscous Hamilton-Jacobi equation.** Let us introduce first two concepts of weak solutions for problem $[(VHJ)+(D)]$ as $p \in (0,\infty)$ and $a \in \mathbb{R}, a \neq 0$.

Definition I.1. *Let $\mu_0 \in \mathcal{M}_b(\Omega)$. A weak solution of problem $[(VHJ)+(D)]$ is a function $u \in C((0,\infty); L^1(\Omega)) \cap L^1(0,T; W_0^{1,1}(\Omega))$ such that $|\nabla u|^p \in L^1(Q_T)$ for all $T > 0$ and satisfying:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = a|\nabla u|^p \quad in \ \ \mathcal{D}'(Q_T), \\ u(t) \underset{t\to 0}{\rightharpoonup} \mu_0 \quad weakly\ in \ \ \mathcal{M}_b(\Omega). \end{cases} \tag{2.1}$$

Definition I.2. *Let $\mu_0 \in \mathcal{M}_b(\Omega)$. A mild solution of problem $[(VHJ)+(D)]$ is a function $u \in C((0,\infty); L^1(\Omega)) \cap L^1(0,T; W_0^{1,1}(\Omega))$ such that $|\nabla u|^p \in L^1(Q_T)$ for all $T > 0$ and satisfying:*

$$u(t) = e^{t\Delta}\mu_0 + a\int_0^t e^{(t-s)\Delta}|\nabla u|^p(s)\, ds. \tag{2.2}$$

The two definitions above are equivalent, thus any mild solution will be called weak solution. We can state now the main results of Chapter I.
The first theorem deals with the case $0 < p < 1$. To our knowledge this case have never been investigated on bounded domains.

Theorem I.1. *Let $a \in \mathbb{R}$, $a \neq 0$, $\mu_0 \in \mathcal{M}_b(\Omega)$ and $0 < p < 1$. Then the problem $[(VHJ)+(D)]$ admits at least one weak solution.*
This solution is unique in the following cases.

a) If $0 < p < \frac{2}{N+1}$ and $\mu_0 \in \mathcal{M}_b(\Omega)$,
b) If $\frac{2}{N+1} \leq p < 1$ and $\mu_0 \in L^q(\Omega)$ with $q > \frac{pN}{2-p}$.

The proof is given in four steps. First, using Schauder fixed point Theorem we prove the local existence of a solution, then we give a regularity result which allows us to extend the local solution to a global in time solution, and finally we show the uniqueness of the solution in the two cases above.

The next case, $1 \le p < \frac{N+2}{N+1}$, has been intensively studied in the whole space $\mathbb{R}^N$ (see [2, 6, 10, 18]) as well as in bounded domains with Dirichlet boundary condition ([1, 11]).

Theorem I.2. *Let $a \in \mathbb{R}$, $a \neq 0$, $\mu_0 \in \mathcal{M}_b(\Omega)$ and $1 \le p < \frac{N+2}{N+1}$. Then the problem $[(VHJ) + (D)]$ admits a unique weak solution.*

The proof follows almost the same steps as in the previous theorem. Since $\xi \longrightarrow |\xi|^p$ is a locally Lipschitz function for $p \ge 1$, the uniqueness result holds. Furthermore the local existence of a solution is obtained thanks to the Banach fixed point Theorem.

We continue the study in the sub-quadratic case and we give some sufficient conditions on the initial data in order to obtain the existence of weak solutions for the problem $[(VHJ) + (D)]$. We denote by

$$q_c = \frac{N(p-1)}{2-p}$$

the critical exponent (which appear also in ([6], p. 343), ([10], p. 2013) and ([11], p. 189) and we notice that

$$q_c \ge 1 \Longleftrightarrow p \ge \frac{N+2}{N+1}.$$

We have the following result:

Theorem I.3. *Let $a \in \mathbb{R}, a \neq 0$ and $\frac{N+2}{N+1} \le p < 2$. If $\mu_0 \in L^q(\Omega)$, where $q > q_c$, then the problem $[(VHJ) + (D)]$ admits a weak solution $u \in C([0,T]; L^q(\Omega)) \cap L^p(0,T; W_0^{1,pq}(\Omega))$ for all $T > 0$. Moreover this solution is unique in the space above.*

The case $a < 0$, requires less restrictive hypothesis on the initial data μ_0, in order to obtain the global existence of a solution for problem $[(VHJ) + (D)]$.

Theorem I.4. *Let $a < 0$, $\frac{N+2}{N+1} \le p < 2$ and $\mu_0 \in L^1(\Omega)$ such that $\mu_0 \ge 0$. Then there exists at least one function $u \in C([0, +\infty); L^1(\Omega))$ which is a weak solution of the problem $[(VHJ) + (D)]$.*

The next theorem is a non-existence result when $p \ge \frac{N+2}{N+1}$ and for initial data in the space $\mathcal{M}_b(\Omega)$ of bounded Radon measures on Ω. We mention that in [10] we can find an analogous result in the whole space $\mathbb{R}^N$.

Theorem I.5. *Let $a < 0$, $p \ge \frac{N+2}{N+1}$, $T > 0$, $x_0 \in \Omega$ and $M > 0$ any positive constant. Then, the problem $[(VHJ) + (D)]$ has no weak solution for the initial data $\mu_0 = M\delta_{x_0}$.*

Finally we deal with the super-quadratic case $p \ge 2$. From Theorem 7.10 in [17] we know that, if $\mu_0 \in C_0^1(\overline{\Omega})$ then the problem $[(VHJ) + (D)]$ has a unique maximal in time, classical solution $u \in C^{0,1}([0,T^*)\times\overline{\Omega})\cap C^{1,2}(Q_{T^*})$ where $T^* \in (0,+\infty]$ is the maximal time existence for the solution u. Moreover if $T^* < \infty$, then:

$$\lim_{t\nearrow T^*}\sup_{x\in\Omega}\left(|u(t,x)| + |\nabla u(t,x)|\right) = \infty. \tag{2.3}$$

As a consequence of the last observation, we notice that u can stay uniformly bounded but does not exist globally in time while:

$$\lim_{t\nearrow T^*}\sup_{x\in\Omega}|\nabla u(t,x)| = +\infty.$$

In this case it is said that gradient blow-up occurs.
If $a > 0,\ p > 2$ and $\mu_0 \in C_0^1(\Omega)$, in [24], P. Souplet has proved that gradient blow-up occurs for problems of type $[(VHJ)+(D)]$, under growth conditions on μ_0. Notice that this phenomenon does not occur for the Cauchy problem on $\mathbb{R}^N$.
In the sequel we prove that for a viscous Hamilton-Jacobi equation with absorption we can have a global existence.

Theorem I.6. *Let $p \geq 2$ and $a < 0$. If the initial data $\mu_0 \in C_0^1(\overline{\Omega})$ and $\mu_0 \geq 0$, then the maximal in time solution u of problem $[(VHJ)+(D)]$ is a global one.*

The last theorem extends Theorem I.6 to less regular initial data. As in [6] we introduce the space:

$$L^1_{+,approx} = \{\mu_0 \in L^1_+(\Omega)|\ \ \exists (u_0^n)_n,\ u_0^n \in C_0^1(\overline{\Omega});\ 0 \leq u_0^n \nearrow \mu_0\}.$$

We have the following result:

Theorem I.7. *Let $p \geq 2$ and $a < 0$. For any $\mu_0 \in L^1_{+,approx}$ the problem $[(VHJ)+(D)]$ has at least one global weak solution u such that, for all $T > 0$:*

$$\begin{cases} u \in C([0,T);L^1(\Omega))\cap L^p(0,T;W_0^{1,p}(\Omega)), \\ \dfrac{\partial u}{\partial t} - \Delta u = a|\nabla u|^p \quad in \quad D'(Q_T), \\ u(0) = \mu_0 \quad\quad in \quad \Omega. \end{cases} \tag{2.4}$$

II. **Cauchy-Neumann problem for viscous Hamilton-Jacobi equation.** As in the previous chapter, we give some existence, uniqueness and regularity results of the solutions of problem $[(VHJ)+(N)]$ depending on the initial data μ_0, the exponent p and the sign of the real parameter a.
Let us briefly summarize the main results of Chapter II:

- (i) When $a \in \mathbb{R}, a \neq 0,\ 0 < p < \frac{N+2}{N+1}$, and μ_0 is a bounded Radon measure, using some fixed point theorems, we show that the problem $[(VHJ)+(N)]$ admits at least one weak solution, moreover if $1 \leq p < \frac{N+2}{N+1}$ then this solution is unique. On the other hand if $\frac{N+2}{N+1} \leq p < 2$ we obtain the existence and uniqueness results as long as the initial data μ_0 is in the Lebesgue space $L^q(\Omega)$, $q > q_c = \frac{N(p-1)}{2-p}$.
- (ii) When $a < 0$ and $\frac{N+2}{N+1} \leq p < 2$ we show that the problem $[(VHJ)+(N)]$ admits at least one weak solution for all $\mu_0 \in L^1(\Omega), \mu_0 \geq 0$.

(*iii*) Finally in the case $a \in \mathbb{R}, a \neq 0$, $p \geq \frac{N+2}{N+1}$ and $\mu_0 = \delta_{x_0}$ (Dirac mass in $x_0 \in \Omega$), we show that $[(VHJ) + (N)]$ has no weak solutions.

III. **Long time behavior for the solutions of Cauchy-Dirichlet problem.** The concept of weak solution has been introduced in Chapter I. Depending on the parameter a and the exponent p we can distinguish among the following results

Theorem III.1. *Let $a < 0$, $p \in (0, 1)$ and $\mu_0 \in \mathcal{M}_b^+(\Omega)$ a positive Radon measure. Then for any weak solution u of problem $[(VHJ) + (D)]$ there exists $T^* > 0$ such that*

$$u(t, x) = 0 \quad \textit{for any} \quad (t, x) \in (T^*, +\infty) \times \Omega. \tag{2.1}$$

This property is called "extinction in finite time of the solution of problem $[(VHJ) + (D)]$".

The proof relies on the results of [8, 9, 18] on the long time behavior of the solutions of the Cauchy problem in the whole space $\mathbb{R}^N$. Indeed, when $a < 0$, after an extension by 0 on $\mathbb{R}^N$ of the initial data, the solution of the Cauchy problem is a super-solution for the Cauchy-Dirichlet problem. Thus, from the extinction in finite time of the solutions of the Cauchy problem we deduce the extinction in finite time of the solutions of the Cauchy-Dirichlet problem.
The following Theorem gives the long time behavior for the solutions of problem $[(VHJ)+(D)]$ as $a < 0$ and $p \geq 1$.

Theorem III.2. *Let $a < 0$, $p \geq 1$ and $\mu_0 \in L^1(\Omega), \mu_0 \geq 0$. Then any weak solution u of problem $[(VHJ) + (D)]$ converges uniformly to 0 as $t \to \infty$.*

This result is a consequence of the fact that the solution $e^{t\Delta}\mu_0$ of the heat equation with initial data μ_0, is a super-solution of the problem $[(VHJ)+(D)]$. Moreover, from the classical results of the heat equation with homogeneous Dirichlet boundary condition (see for example Lemma 3, p. 25 in [23]) we know that:

$$\|e^{t\Delta}\mu_0\|_\infty \leq C(1 + t^{-\frac{N}{2}})e^{-\lambda t}\|\mu_0\|_1, \quad \text{for all} \;\; t \in (0, +\infty),$$

where λ is the first eigenvalue of the Laplacien in Ω with Dirichlet boundary condition. We deduce that, the weak solution u of problem $[(VHJ) + (D)]$ satisfies:

$$\|u(t)\|_\infty \leq C(1 + t^{-\frac{N}{2}})e^{-\lambda t}\|\mu_0\|_1, \quad \text{for all} \;\; t \in (0, +\infty). \tag{2.2}$$

Which gives the decreasing rate of the solution u. In particular, u converges uniformly to 0 as $t \to \infty$. Thus, the proof of Theorem III.2 is achieved. □

We continue the study on the long time behavior with the case $a > 0$, $p \in [1, 2)$.

Theorem III.3. *Let $a \in \mathbb{R}, a \neq 0$, $p \in (1, 2)$ and $\mu_0 \in C_0(\overline{\Omega})$. Then the global solution u of problem $[(VHJ) + (D)]$ converges to 0, uniformly in $\overline{\Omega}$, as $t \to \infty$.*

The proof of Theorem III.3 relies on the one hand on the LaSalle Invariance Principle and on the other hand on the convergence of the trajectories to the equilibrium points when there exists a strictly Liapunov function for the dynamical system generated by the solutions of $[(VHJ) + (D)]$. Moreover this result can be extended to initial data less regular as in Theorems I.2 and I.3.

IV. **Long time behavior for the solutions of Cauchy-Neumann problem.** In the last chapter we reconsider the problem $[(VHJ)+(N)]$ with Ω a bounded and convex open set, and we give some existence and uniqueness results of the solutions when the initial data is a continuous function in $\overline{\Omega}$. Then we study the large time behavior of the solutions according to the exponent p. These results rely on some remarkable estimates for the gradient of the solutions of problem $[(VHJ)+(N)]$, obtained by using a Bernstein technique. Finally we consider the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + |\nabla u|^p = 0 \text{ in } \Omega \times (0,+\infty); \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad (0,+\infty) \times \partial\Omega; \\ u(0) = +\infty \quad \text{in} \quad D; \\ u(0) = 0 \quad \text{in} \quad \Omega \setminus \overline{D}; \end{cases} \tag{2.1}$$

where $p \in (1,+\infty)$, Ω and D are two open sets with smooth boundary such that $D \subset \overline{D} \subset \Omega$ and Ω is convex. This problem is related to the large deviation theory for some diffusion process.

Before stating the main results we need to introduce the following notation. Let u be a function in $C(\overline{Q_\infty})$. For any $t \geq 0$ denote by:

$$M(t) = \max_{x \in \overline{\Omega}} u(t,x) \tag{2.2}$$

and

$$m(t) = \min_{x \in \overline{\Omega}} u(t,x), \tag{2.3}$$

Theorem IV.1. *Consider $a \in \mathbb{R}, a \neq 0$, $p > 0$ and $\mu_0 \in C(\overline{\Omega})$, where Ω is a bounded and convex open set. Then, the problem $[(VHJ)+(N)]$ admits a unique solution:*

$$u \in C(\overline{Q_T}) \cap C^{1+\delta/2, 2+\delta}(\overline{Q_{\tau,T}})$$

for any $T > 0$ and $\tau \in (0,T)$. Moreover, we have:

$$t \to M(t) \text{ is a decreasing function in } \mathbb{R}, \tag{2.4}$$

$$t \to m(t) \quad \text{is a non-decreasing function in } \mathbb{R}, \tag{2.5}$$

$$\|\nabla u(t)\|_\infty \leq \left(\frac{1}{2}\right)^{1/2} (M(s)-m(s))(t-s)^{-\frac{1}{2}} \text{ for all } t > s \geq 0, \tag{2.6}$$

and for $p \neq 1$

$$\|\nabla u(t)\|_\infty \leq \left(\frac{\max\{p,2\}}{ap|1-p|}\right)^{1/p} (M(s)-m(s))^{1/p}(t-s)^{-1/p} \text{ for all } t > s \geq 0. \tag{2.7}$$

For the proof we shall use the Bernstein technique. This method can be found in [10, 11, 18] and [22], where similar estimates to (2.6) and (2.7) are obtained for the Cauchy problem in $\mathbb{R}^N$.

In the next part we shall analyze the large time behavior of the solutions of $[(VHJ)+(N)]$ as $a < 0$, $p > 1$ and the initial data μ_0 is a continuous and positive function. Thus, we find an universal bound for the gradient of the solution, which will be very useful in the proof of Theorem IV.4.

Theorem IV.2. *Let Ω be a bounded and convex open set, $a < 0$ and $p > 1$. Let $\mu_0 \in C^+(\overline{\Omega})$ and u the unique solution of problem $[(VHJ) + (N)]$ given by Theorem IV.1. Then u satisfies the following estimations:*

$$\|\nabla u^{(p-1)/p}(t)\|_\infty \le C(p,\Omega)\|u_0\|_1^{(p-1)/p} t^{-(p(N+1)-N)/2p}, \tag{2.8}$$

and

$$\|\nabla u^{(p-1)/p}(t)\|_\infty \le |a|^{-1/p}\frac{(p-1)^{(p-1)/p}}{p} t^{-1/p}. \tag{2.9}$$

The following result gives the asymptotical behavior of solutions of problem $[(VHJ) + (N)]$ depending on exponent p.

Theorem IV.3. *Consider $a \in \mathbb{R}, a \neq 0$, $p > 0$ and Ω a bounded and convex domain. Let $\mu_0 \in C(\overline{\Omega})$ and denote by u the solution of problem $[(VHJ)+(N)]$ corresponding to μ_0. Then:*

i) If $p \in (0,1)$, the extinction of the gradient of u in finite time occurs, in other words: there exists $T^* \in [0,+\infty)$ and $c \in \mathbb{R}$ such that:

$$u(t,x) \equiv c \textit{ for all } t \ge T^* \textit{ and } x \in \overline{\Omega},$$

ii) If $p \in [1,+\infty)$, then $u(t,\cdot)$ converges uniformly on $\overline{\Omega}$ to a constant, as $t \to \infty$.

The proof of Theorem IV.3 follows the same ideas as in [9]. In this paper, the authors investigate the large time behavior for the Cauchy problem in the whole space $\mathbb{R}^N$ and for initial data periodic functions. We mention that the key arguments of the proof are the estimates (2.6) and (2.7) above, moreover the results can be extended to less irregular initial data as in Chapter II.

Finally we study the problem (2.1) whose origin comes from the large deviation theory. We recall that the same problem with Dirichlet boundary condition has been already analyzed in [11] for $p > 1$ and $a < 0$. It is suitable to introduce first the following definition:

Definition IV.1. *A weak solution of problem (2.1) is a positive function $u \in C^{1,2}(\overline{Q_{\tau,T}})$ which satisfies for all $0 < \tau < T < \infty$:*

$$\begin{cases} u_t - \Delta u + |\nabla u|^p = 0 \textit{ in } Q_{\tau,T}, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \textit{on } \Gamma_{\tau,T}, \end{cases} \tag{2.10}$$

and the initial condition:

$$\lim_{t\searrow 0} u(t,\cdot) = 0 \textit{ uniformly on any compact subset of } \overline{\Omega}\setminus\overline{D} \tag{2.11}$$

$$\lim_{t\searrow 0} u(t,\cdot) = +\infty \textit{ uniformly on any compact subset of } D. \tag{2.12}$$

Theorem IV.4. *Suppose that $p \in (1,2)$. Let Ω, D be two open domains of $\mathbb{R}^N$ with smooth boundary such that Ω is convex and $\overline{D} \subset \Omega$. Then the problem (2.1) admits a unique solution in the sense of Definition IV.1.*

The existence proof uses some properties of order preserving semigroups, which are developed in [11]. The convexity of Ω plays an essential role, more precisely for the proof we shall need the gradient estimations obtained in Theorem IV.2.

References

[1] N. Alaa. Solutions faibles d'équations paraboliques quasilinéaires avec données initiales mesure. *Ann. Math. Blaise Pascal*, 3:1–15, 1996.

[2] L. Amour and M. Ben-Artzi. Global existence and decay for a viscous Hamilton-Jacobi equation . *Nonlinear Anal*, 31:621–628, 1998.

[3] G. Barles. *Solution de viscosité des équations de Hamilton-Jacobi*, volume 17 of *Mathématiques et Applications.* Springer, 1994.

[4] M. Ben-Artzi. Global existence and decay for a nonlinear parabolic equation. *Nonlinear Anal.*, 19:763–768, 1992.

[5] M. Ben-Artzi and H. Koch. Decay of mass for semilinear parabolic equation . *Comm. Partial Differential Equations*, 25(5-6):869–881, 1999.

[6] M. Ben-Artzi, Ph. Souplet, and F.B.Weissler. The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces . *J. Math. Pures. Appl.*, 81:343–378, 2002.

[7] S. Benachour and S. Dabuleanu. The mixed Cauchy-Dirichlet problem for a viscous Hamilton-Jacobi equation . *Advances Diff. Equations*, 8(12):1409-1452, 2003.

[8] S. Benachour, Ph. Laurençot, and D. Schmitt. Extinction and decay estimates for viscous Hamilton-Jacobi equations in $\mathbb{R}^N$. *Proc. Amer. Math. Soc.*, 130(4):1103–1111, 2002.

[9] S. Benachour, Ph. Laurençot, D. Schmitt, and Ph. Souplet. Extinction and non-extinction for viscous Hamilton-Jacobi equation in $\mathbf{R^N}$. *Asymptot. Anal.*, 31(3-4):229–246, 2002.

[10] S. Benachour and Ph. Laurencot. Global solutions to viscous Hamilton-Jacobi equations with irregular initial data . *Comm. Partial Differential Equations*, 24:1999–2021, 1999.

[11] M.G. Crandall, P.-L.Lions, and P.E. Souganidis. Maximal solutions and universal bounds for some partial differential equations of evolution . *Arch.Rational Mech.Anal.*, 105(2):163–190, 1989.

[12] S. Dabuleanu. The Cauchy-Neumann problem for a viscous Hamilton-Jacobi equation . *In press at J.Evol.Equations*, 2003.
Problèmes aux limites pour des équations de Hamilton-Jacobi avec viscuosité . *Prépublications IECN*, 2003.

[13] L.C. Evans and H. Ishii. A PDE approach to some asymptotic problems concerning random differential equations with small noise intensities . *Ann. Inst. H. Poincaré, Analyse non Linéaire*, 2:1–20, 1985.

[14] L.C. Evans and P.E. Souganidis. A PDE approach to geometric optics for certain reaction di ffusion equations . *Ind.U.Math.J.*, 38:773–797, 1989.

[15] W.H. Fleming and R.W. Rishel. *Deterministic and stochastic optimal control*. Springer, Berlin, 1975.

[16] W.H. Fleming and P.E. Souganidis. A PDE viscosity solution approach to some problems of large deviation . *Ann.Scuola Norm. Sup.*, 13:171–192, 1986.

[17] A. Friedman. *Partial differential equations of parabolic type* . Prentice-Hall, Inc., 1964.

[18] B. Gilding, M. Guedda, and R. Kersner. The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$. *J. Math.Anal.Appl.*, 284:733–755, 2003.

[19] M. Kardar, G. Parisi, and Y.-C. Zang. Dynamic scaling of growing interfaces . *Phys. Rev. Lett.*, 56:889–892, 1986.

[20] J. Krug and H. Spohn. Universality classes for deterministic surface growth . *Phys. Rev. A.*, 38:4271–4283, 1988.

[21] Ph. Laurençot and Ph. Souplet. On the growth of mass for a viscous Hamilton-Jacobi equation . *J.d'Analyse Math.*, 89:367–383, 2003.

[22] P.L. Lions. Résolution de problèmes elliptiques quasilinéaires . *Arch. Rati. Mech. Anal.*, 74:335–353, 1980.

[23] F. Rothe. *Global solutions of reaction-diffusion systems* , volume 1072. Springer-Verlag, 1984.

[24] Ph. Souplet. Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions . *Diff.Int. Equation*, 15(2):237–256, 2002.